Interpolation of Forward Rates in the LIBOR Market Model

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A dissertation submitted to the Faculty of Commerce, University of Cape Town, in partial fulfilment of the requirements for the degree of Master of Philosophy.

July 16, 2019

MPhil in Mathematical Finance,
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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy to the University of Cape Town. It has not before been submitted for any degree or examination.

Signed by candidate

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July 16, 2019

Abstract

Since its development in 1997, the LIBOR market model has gained widespread use in interest rate modelling, largely owing to its consistency with the Black futures formula for pricing interest rate caps and floors. From its original construction(s), the LIBOR market model specifies a discrete set of forward rates that correspond to a fixed tenor structure, e.g. market tenors. This implies the pricing of interest rate contingent claims is restricted to claims with cashflow dates that coincide with the fixed tenor structure. In this light, several interpolation schemes have been suggested to handle the pricing restrictions, however at the cost of introducing possible arbitrage opportunities. The present dissertation studies four such interpolation schemes, paying particular attention to arbitrage-free interpolation schemes: Piterbarg deterministic interpolation, Schlögl deterministic interpolation, Schlögl stochastic interpolation, and Beveridge-Joshi stochastic interpolation.

Acknowledgements

I wish to thank Assoc. Prof. Thomas McWalter, my research supervisor, for his patience, guidance and constructive feedback offered throughout the progress of this dissertation. Special thanks are due to Assoc. Prof. Peter Ouwehand for his immense knowledge, and Assoc. Prof. David Taylor for making this research possible.

Finally, I wish to thank my parents for their continuous moral support over the course of my studies.

"The beginner should not be discouraged if he finds he does not have the prerequisites for reading the prerequisites." — Paul Halmos

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Chapter 1

Introduction

The market valuation of interest rate contingent claims, such as interest rate caps and floors, has long been carried out under the assumption that forward rates follow a lognormal (martingale) process with zero drift; a practice usually accredited to Black (1976). This market practice ignored the arbitrage relationships between forward rates of different tenors, raising much scepticism in the academic community. Moreover, it was long known that any term structure model with lognormal forward rates under a single measure admits arbitrage opportunities, which then propelled research into arbitrage-free term structure models that were consistent with the lognormal assumption (Brace, Gatarek and Musiela, 1997).

Prior to the LIBOR market model, the evolution of all interest rates was described by a single instantaneous rate called the short rate. Direct modelling of the short rate presented several limitations; the approach was difficult to calibrate to the market, and lacked the flexibility of modelling observable forward rates directly. Heath, Jarrow and Morton (1992) (HJM) then introduced a general term structure framework to model the evolution of instantaneous forward rates instead; constructing a model that was naturally calibrated to an initial forward rate curve, a feature prior (short rate) models lacked. The HJM framework, though, suffered a serious drawback in that instantaneous forward rates were merely mathematical constructs and not observable in the market (Schlögl, 2002). This led to the development of the LIBOR market model by Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997), and Jamshidian (1997), in which forward LIBOR rates were characterised as lognormal martingales under forward measures corresponding to their maturities.

The purpose of arbitrage pricing theory in the context of term structure modelling is to price all interest rate derivative contracts upon specifying zero coupon bond prices (Brace, Gatarek and Musiela, 1997). The LIBOR market model specifies a discrete set of forward rates that correspond to a fixed tenor structure (set of maturities), and thus limits valuation to interest rate contingent claims with tenor cashflows, i.e. cashflows lying inside the fixed tenor structure. To handle the pricing restrictions, several interpolation methods have been suggested in practice to obtain interest rates corresponding to non-tenor cashflow dates. The problem with these methods is that they have no regard for possible arbitrage opportunities introduced as a result of the interpolation.

Instead, in this dissertation we place more emphasis on arbitrage-free interpolation schemes due to Schlögl (2002) and Beveridge and Joshi (2012), but we also consider the intuitive Piterbarg interpolation scheme. More specifically, our aim is to provide a comprehensive review and comparison of the following four interpolation schemes: Piterbarg deterministic interpolation, Schlögl deterministic interpolation, Schlögl stochastic interpolation, and Beveridge-Joshi stochastic interpolation. In addition to the three desirable properties suggested by Beveridge and Joshi (2012), we also consider the smoothness of interpolated rates as a criterion for comparison.

This dissertation is organised as follows. In Chapter 2 we review basic and relevant stochastic calculus concepts that will be utilised throughout this dissertation. In Chapter 3 we present a rigorous construction of the LIBOR market model, following the original forward measure approach of Musiela and Rutkowski (1997). The spot LIBOR market model is also constructed, following a much simpler approach than the original Jamshidian (1997) approach. In Chapter 4 we define our problem statement more clearly and discuss the above-mentioned interpolation schemes in detail. In Chapter 5 we consider the modelling of volatility and correlation of forward LIBOR rates. Caplet pricing within the LIBOR market model is discussed briefly, followed by a section on the simulation of the LIBOR market model. In Chapter 6 we present numerical results concerning the implementation of the four interpolation schemes. Finally, in Chapter 7 we present our conclusions.

Chapter 2

Stochastic Calculus Review

A sufficient knowledge of stochastic calculus and measure-theoretic probability theory is assumed. In this section we review basic and relevant stochastic calculus concepts that will be utilised throughout this dissertation. Much of the material here is based on the following books: Rogers and Williams (2000), Protter (2004).

Definition 2.1 (Stochastic process)**.** Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a *stochastic process* X is a map $X : [0, \infty) \times \Omega \to \mathbb{R}$, $(t, \omega) \mapsto X_t(\omega)$ such that X_t is $\mathcal{F}$-measurable for all $t \geq 0$. In this way, a stochastic process X may also be viewed as a family $(X_t)_{t \geq 0}$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. We shall write *process* to mean stochastic process.

Definition 2.2 (Sample path)**.** A *sample path* of a process X is a particular realisation of the process. That is, for a particular $\omega \in \Omega$, a *sample path* corresponds to the function $X(\cdot, \omega) : [0, \infty) \to \mathbb{R}, t \mapsto X_t(\omega)$. A process X is then *(left/right-)continuous* if its sample paths are (left/right-)continuous almost surely.

Sometimes it is more desirable to identify a process X with its sample paths, in which case it becomes a random function $X : \Omega \to \mathbb{R}^{[0,\infty)}$. Certain regularity conditions, such as measurability or continuity, are then usually imposed on X.

Definition 2.3 (Uniform integrability)**.** A stochastic process X is uniformly integrable if the family $(X_t)_{t \geq 0}$ of random variables is uniformly integrable, i.e. if

$$\lim_{K \to \infty} \mathbb{E}\left[|X_t| \mathbb{1}_{\{|X_t| > K\}}\right] = 0.$$

Definition 2.4 (Filtration)**.** A *filtration* $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is an increasing family of σ-algebras, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s \leq t$. The tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is then called a *filtered probability space*. A process X is said to be *adapted* to $\mathbb{F}$ if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

Definition 2.5 (Usual conditions)**.** A filtration $\mathbb{F}$ satisfies the *usual conditions* if

(i) $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets,

(ii) $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \geq 0$.

Definition 2.6 (Natural filtration)**.** The *natural filtration* of a process X (denoted $\mathbb{F}^X$) is the smallest filtration to which X is adapted, i.e. $\mathcal{F}_t^X := \sigma(X_s : s \leq t)$.

Definition 2.7 (Martingale)**.** An adapted process X is a *martingale* (w.r.t. the filtration $\mathbb{F}$) if

(i) $\mathbb{E}[|X_t|] < \infty$ for all $t \geq 0$,

(ii) $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ for all $s \leq t$.

When the underlying probability measure is not clear in (i) and (ii), we shall write X is a $\mathbb{P}$-martingale to mean $\mathbb{E} = \mathbb{E}^{\mathbb{P}}$. The set of martingales X is denoted by $\mathcal{M}$. An important subspace of $\mathcal{M}$ is c$\mathcal{M}_0$, consisting of continuous martingales X with $X_0 = 0$ a.s.

Definition 2.8 (Brownian motion)**.** An adapted process W is called a *d-dimensional Brownian motion* if

(i) $W_0 = 0$ a.s.,

(ii) W is continuous,

(iii) $W_t - W_s$ is independent of $\mathcal{F}_s$ for all $s \leq t$,

(iv) $W_t - W_s$ is Gaussian with mean $\mathbf{0}$ and variance $(t-s)I_d$ for all $s \leq t$.

Definition 2.9 (Finite variation)**.** An adapted process A has *finite variation* if almost all its sample paths are locally of bounded variation. The set of continuous finite variation processes A with $A_0 = 0$ a.s. is denoted by cFV_0.

Definition 2.10 ($u.c.p.$)**.** A sequence $\{X^n\}$ of processes converges uniformly on compacts in probability to a limit process X, written $X^n \to X$ $u.c.p.$, if for all $t > 0$,

$$\sup_{s\leq t}|X_s^n - X_s| \to 0 \text{ in probability as } n \to \infty.$$

Theorem 2.11 (Quadratic variation)**.** *Let $X \in c\mathcal{M}_0$ and consider a sequence $\{\langle X\rangle^{(n)}\}$ of processes defined by*

$$\langle X\rangle_t^{(n)} := \sum_{i\geq 1} \left(X_{t\wedge i2^{-n}} - X_{t\wedge(i-1)2^{-n}}\right)^2, \ t \geq 0.$$

Then there exists a continuous finite variation increasing process, denoted by $\langle X\rangle$, such that

$$\langle X\rangle^{(n)} \to \langle X\rangle \ u.c.p.$$

The angle-brackets process $\langle X\rangle$ is called the quadratic variation *of X.*

Definition 2.12 (Quadratic covariation)**.** Let $X, Y \in c\mathcal{M}_0$. Then the *quadratic covariation* between X and Y, denoted by $\langle X, Y \rangle$, is the continuous finite variation process defined through the polarisation identity

$$\langle X, Y \rangle := \frac{1}{4}(\langle X + Y \rangle - \langle X - Y \rangle).$$

Note, $\langle X, X \rangle = \langle X \rangle$.

Definition 2.13 (Previsible)**.** The *previsible σ-algebra*, denoted $\mathcal{P}$, is the σ-algebra on $[0, \infty) \times \Omega$ generated by left-continuous adapted processes. A process H is said to be *previsible* if it is $\mathcal{P}$-measurable. Now let $X \in c\mathcal{M}_0$. An important class of previsible processes for defining the stochastic integral w.r.t. X is $L^2(X)$, defined as the set of (equivalence classes of) previsible processes H with the property

$$\mathbb{E}\left[\int_0^\infty H_s^2 d\langle X \rangle_s\right] < \infty \text{ a.s.}$$

The processes in $L^2(X)$ are the admissible previsible integrands.

The following result of Kunita-Watanabe provides a means to compute the quadratic variation and covariation of stochastic integrals. More generally, this result can be used to characterise the stochastic integral, yielding the Kunita-Watanabe definition of the stochastic integral.

Theorem 2.14 (Kunita-Watanabe)**.** *Let $X \in c\mathcal{M}_0$ and $H \in L^2(X)$. Then the* stochastic integral *of H w.r.t. X, denoted $\int H dX$, satisfies*

$$\left\langle \int H dX, Y \right\rangle = \int H d\langle X, Y \rangle,$$

for all $Y \in c\mathcal{M}_0$.

The processes $H = (H^1, ..., H^d)$ and $X = (X^1, ..., X^d)$ may also be $\mathbb{R}^d$-valued, in which case we say $H \in L^2(X)$ and X is a continuous martingale if, for $i = 1, ..., d$, $H^i \in L^2(X^i)$ and X^i is a continuous martingale. Naturally, the stochastic integral is then generalised as

$$\int H \cdot dX := \sum_{i=1}^d \int H^i dX^i.$$

Theorem 2.15 (Itô's formula)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be twice continuously differentiable and $X = (X^1, ..., X^d)$ be an $\mathbb{R}^d$-valued continuous martingale. Then*

$$df(X_t) = \sum_{i=1}^d \frac{\partial f}{\partial x^i}(X_t) dX_t^i + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x^i \partial x^j}(X_t) d\langle X^i, X^j \rangle_t.$$

Chapter 3

The LIBOR Market Model

The aims of this chapter are twofold: First, to present the LIBOR market model in a clear and concise manner; and second, to define our problem along with four interpolation schemes. We begin by introducing some notation and definitions that will be utilised throughout this dissertation. The following definitions are a slight modification of those provided by Filipovic (2009).

Definition 3.1 (Bond)**.** A *zero-coupon bond* with maturity T (also called T-*bond*) is a contract that pays a unit of currency at time T. The time $t \leq T$ value of this contract is denoted by $P(t, T)$.

Later in this chapter we introduce a classification of bonds into four categories. This classification looks at when a bond matures (i.e. T), and how far into the future that maturity is (i.e. $T - t$).

Definition 3.2 (LIBOR)**.** Let $t \leq T < U$. Then the *forward LIBOR rate* prevailing at time t for the period $[T, U]$, denoted $L(t, T, U)$, is defined as

$$L(t, T, U) := \frac{1}{\delta(T, U)} \left(\frac{P(t, T)}{P(t, U)} - 1 \right), \tag{3.1}$$

where $\delta(T, U)$ is the year fraction (in accordance with market conventions) between expiry T and maturity U.

When the maturity U is clear from context, we will denote the forward LIBOR rate prevailing at time t for the period $[T, U]$ simply as $L(t, T)$. We ignore any day-count conventions, and take $\delta(T, U)$ to be the difference $U - T$ expressed in years. Beyond the reset date T, $L(t, T)$ is assumed to remain constant so that $L(t, T) = L(t \wedge T, T)$.

Definition 3.3 (Forward rate)**.** The *instantaneous forward rate* prevailing at time t for the maturity T, denoted $f(t, T)$, is defined as

$$f(t, T) := \lim_{U \downarrow T} L(t, T, U) = -\frac{\partial}{\partial T} \log P(t, T).$$

Suppose now a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}^{T_N})$ equipped with a d-dimensional Brownian Motion W^{T_N}, where $\mathbb{F}$ is the natural filtration generated by W^{T_N}, i.e. $\mathbb{F} = \mathbb{F}^{W^{T_N}}$. When we want to be explicit about the Brownian motion W^{T_N}, we shall write this space as $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}^{T_N}, W^{T_N})$. Naturally, we assume the filtration $\mathbb{F}$ satisfies the usual conditions.

Let $\mathbb{T} := \{T_0, ..., T_N\}$ denote a discrete-tenor structure associated with the year fractions $\delta\mathbb{T} := \{\delta_0, ..., \delta_{N-1}\}$, where $\delta_i := \delta(T_i, T_{i+1}) = T_{i+1} - T_i$. In practice, the tenor structure $\mathbb{T}$ is typically set to match standard market tenors. We assume the existence of a bond market $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}^{T_N}, \mathbb{B})$, where $\mathbb{B} = \{P(\cdot, T_i)\}$, such that the following assumptions hold (Musiela and Rutkowski, 1997):

Assumption 3.4. For any maturity $T \in \mathbb{T}$, the bond price process $P(\cdot, T)$ is a continuous process satisfying $P(t, T) > 0$ for all $t \in [0, T]$.

Assumption 3.5. For any maturity $T \in \mathbb{T}$, the T_N-forward process

$$\frac{P(t, T)}{P(t, T_N)}, \ t \in [0, T],$$

is a $\mathbb{Q}^{T_N}$-martingale.

Assumption 3.6. For any maturity $T \in \mathbb{T}$, the bond price process $P(\cdot, T)$ satisfies $P(t, T) \leq 1$ for all $t \in [0, T]$.

Assumption 3.4 ensures that the bond price processes dealt with are sufficiently regular for stochastic integration. Assumptions 3.5–3.6 are no-arbitrage conditions; 3.5 precludes any arbitrages between bonds while 3.6 precludes arbitrage between bonds and cash.

Convention: It is a common convention to assume a value of one for the empty product and zero for the empty sum. That is, given a sequence $\{x_i\}$, if $k < j$ then $\prod_{i=j}^{k} x_i := 1$ and $\sum_{i=j}^{k} x_i := 0$.

3.1 Change of Numéraire

In this section we review Girsanov's theorem, along with other crucial results that will prove useful in the construction of the LIBOR market model. Much of the material here is based on the following books: Rogers and Williams (2000), Protter (2004), Björk (2009).

Definition 3.7 (Equivalent measures)**.** Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega, \mathcal{F})$. Then $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent*, written $\mathbb{Q} \sim \mathbb{P}$, if they agree on null sets, i.e. if $\mathbb{P}(A) = 0$ iff $\mathbb{Q}(A) = 0$ for all $A \in \mathcal{F}$.

Theorem 3.8 (Bayes' formula)**.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent probability measures on $(\Omega, \mathcal{F}, \mathbb{F})$ with density $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. Then for any $\mathbb{Q}$-integrable random variable X and sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$, we have*

$$\mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{G}] = \frac{\mathbb{E}^{\mathbb{P}}[X\,Z \mid \mathcal{G}]}{\mathbb{E}^{\mathbb{P}}[Z \mid \mathcal{G}]},\ \mathbb{Q}\text{-}a.s.$$

Proof. The proof can be found in Björk (2009). □

Thus with Bayes' formula, we can express the relationship between conditional expectations under two equivalent measures.

Definition 3.9 (Doléans exponential)**.** Let $X \in c\mathcal{M}_0$. Then the *Doléans exponential* of X, denoted $\mathcal{E}(X)$, is the process defined by

$$\mathcal{E}_t(X) := e^{X_t - \frac{1}{2}\langle X\rangle_t}.$$

Theorem 3.10 (Doléans)**.** *Let $X \in c\mathcal{M}_0$ be such that $\mathcal{E}(X)$ is a martingale. Then the process $Z = Z_0\mathcal{E}(X)$ is the unique continuous martingale satisfying*

$$Z_t = Z_0 + \int_0^t Z_s dX_s.$$

Proof. The proof can be found in Rogers and Williams (2000). □

The stochastic differential equations (SDEs) dealt with in this dissertation will be of the form prescribed in *Thm.* 3.10, hence the importance of the Doléans exponential. Furthermore, the Doléans exponential provides an appropriate transformation for changing measures as shown by the next theorem.

Theorem 3.11 (Girsanov)**.** *Let W be a d-dimensional $\mathbb{P}$-Brownian motion on $(\Omega, \mathcal{F})$, and let $H \in L^2(W)$ be an $\mathbb{R}^d$-valued process. Define the process Z by $Z_t = \mathcal{E}_t(\int H dW)$. If Z is a uniformly integrable martingale, then a new measure $\mathbb{Q}\,(\sim \mathbb{P})$ may be defined by the density*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := Z_\infty.$$

The process $W_t - \int_0^t H_s ds$ is then a d-dimensional $\mathbb{Q}$-Brownian motion.

Proof. The proof can be found in Rogers and Williams (2000). □

It is often difficult to directly check uniform integrability of the Doléans exponential Z in Girsanov's theorem. Fortunately, Novikov's criterion provides a sufficient condition that is easier to check.

Theorem 3.12 (Novikov's criterion)**.** *Let* $X \in c\mathcal{M}_0$. *If* $\mathbb{E}[e^{\frac{1}{2}\langle X\rangle_\infty}] < \infty$, *then* $\mathcal{E}(M)$ *is a uniformly integrable martingale.*

Proof. The proof can be found in Protter (2004). □

Note, however, that this criterion is not equivalent to uniform integrability.

Definition 3.13 (Numéraire)**.** A *numéraire* is any asset N whose price process is adapted and satisfies $N_t > 0$ for all $t \geq 0$ a.s.

As we shall see shortly, a numéraire acts as a unit of account to denominate assets within a particular market $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{M})$. We restrict our attention to the previously mentioned bond market $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}^{T_N}, \mathbb{B})$, where $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}^{T_N})$ is a filtered probability space and $\mathbb{B} = \{P(\cdot, T_i)\}$. Often we will simply identify the bond market with $\mathbb{B}$.

Definition 3.14 (Martingale measure)**.** Given a market $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{M})$ and numéraire N, a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is an *equivalent martingale measure* (EMM) associated with numéraire N if

(i) $\mathbb{Q} \sim \mathbb{P}$,

(ii) $\left(\frac{X_t}{N_t}\right)_{t\geq 0}$ is a $\mathbb{Q}$-martingale for all $X \in \mathbb{M}$.

The process $\left(\frac{X_t}{N_t}\right)_{t\geq 0}$ is then called the deflated price process of X.

Theorem 3.15 (Geman *et al.* (1995))**.** *Let* N *and* X *be numéraires, and fix* $T > 0$ *such that* N_T *and* X_T *are both defined. Let* $\mathbb{Q}$ *be the EMM associated to N. Then the measure* $\mathbb{Q}^X$ *on* $(\Omega, \mathcal{F}_T)$, *defined by the density*

$$\frac{d\mathbb{Q}^X}{d\mathbb{Q}} := \frac{X_T/X_0}{N_T/N_0},$$

is an EMM associated to X.

Proof. The proof is a straightforward application of Bayes' formula (*Thm.* 3.8) and can be found in Geman *et al.* (1995). □

Note, the density is given by normalised asset ratios at the terminal date T. Similar to Girsanov's theorem, *Thm.* 3.15 also provides an appropriate transformation (but in terms of numéraires) for changing measures.

Definition 3.16 (Forward measure)**.** Fix $T \in \mathbb{T}$. The *T-forward measure* on $(\Omega, \mathcal{F}_T)$, denoted by $\mathbb{Q}^T$, is the EMM associated with the T-bond $P(\cdot, T)$ as a numéraire.

3.2 Construction: LIBOR Market Model

The construction of the LIBOR market model is detailed in this section. We begin with the following definition of the LIBOR market model by Björk (2009). For ease of notation, we write $L(\cdot, T_i)$ to denote the forward LIBOR rate $L(\cdot, T_i, T_{i+1})$.

Definition 3.17 (LIBOR market model)**.** The family $\{L(\cdot, T_i)\}$ of forward LIBOR rates is called a *LIBOR market model* if for $i = 1, ..., N-1$,

$$dL(t, T_i) = L(t, T_i)\sigma(t, T_i) \cdot dW_t^{T_{i+1}},\ t \in [0, T_i],$$

where $W^{T_{i+1}}$ is a d-dimensional $\mathbb{Q}^{T_{i+1}}$-Brownian motion such that the family $\{W^{T_{i+1}}\}$ is independent, and $\sigma(\cdot, T_i)$ is an $\mathbb{R}^d$-valued bounded deterministic volatility function.

The volatility functions $\{\sigma(\cdot, T_i)\}$ are the component row functions of the square root of σ, where σ is the instantaneous covariance matrix for the family $\{\log L(\cdot, T_i)\}$ of random variables. That is, the instantaneous covariance between $\log L(\cdot, T_i)$ and $\log L(\cdot, T_j)$ is given by $\sigma_{i,j}(t) = \sigma(t, T_i) \cdot \sigma(t, T_j)$. Hence, the family $\{\sigma(\cdot, T_i)\}$ contains essential information regarding the correlation structure of the LIBOR market model. Intuitively, $\sigma(t, T_i)$ describes the responsiveness of $L(\cdot, T_i)$ to the d random shocks represented by the d-dimensional Brownian motion $W^{T_{i+1}}$. An alternative formulation will be discussed in Section 3.3.

Def. 3.17 is consistent with the market convention that forward LIBOR rates follow a lognormal martingale process, hence the use of Black's futures formula is well justified in this framework. It is important to note, however, that each forward LIBOR rate $L(\cdot, T_i)$ is a lognormal martingale only under its own forward measure $\mathbb{Q}^{T_{i+1}}$ instead of simultaneously under a single measure. The boundedness restriction placed on $\sigma(\cdot, T_i)$ ensures that Novikov's criterion is satisfied over any finite time horizon, hence validating the application of Girsanov's theorem in the proofs below. That $\sigma(\cdot, T_i)$ is deterministic ensures the lognormality of $L(\cdot, T_i)$.

Later, once we have constructed the LIBOR market model, we will introduce the spot LIBOR measure by Jamshidian (1997), and show that under this single measure the forward LIBOR rates are no longer lognormal martingales. Our concern with the spot LIBOR measure stems from its advantage over forward measures in regard to the simulation of the LIBOR market model. We will return to this issue in Section 5.4.

It is not clear from *Def.* 3.17 whether there actually exists a LIBOR market model. *Thm.* 3.18 below demonstrates the existence using the original forward measure approach of Musiela and Rutkowski (1997).

Theorem 3.18 (Musiela and Rutkowski (1997))**.** *Suppose the family $\{\sigma(\cdot, T_i)\}$ satisfies the conditions in Def. 3.17. Then there exists a LIBOR market model on the bond market* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}^{T_N}, W^{T_N}, \mathbb{B})$.

Proof. (Musiela and Rutkowski, 1997). We present a constructive proof of the LIBOR market model using (backward) induction. (Case: $N-1$) First we define the forward LIBOR rate corresponding to expiry T_{N-1} as the process satisfying

$$
\begin{aligned}
dL(t, T_{N-1}) &= L(t, T_{N-1})\sigma(t, T_{N-1}) \cdot dW_t^{T_N},\ t \in [0, T_{N-1}], \\
L(0, T_{N-1}) &= \frac{1}{\delta_{N-1}} \left(\frac{P(0, T_{N-1})}{P(0, T_N)} - 1 \right).
\end{aligned}
$$

(Case: $N-2$) Next we define the forward LIBOR rate corresponding to expiry T_{N-2}. For this we need to consider the one-period forward process

$$
\begin{aligned}
d\left(\frac{P(t, T_{N-1})}{P(t, T_N)} \right) &= d(1 + \delta_{N-1} L(t, T_{N-1})) \\
&= \delta_{N-1} dL(t, T_{N-1}) \\
&= \delta_{N-1} L(t, T_{N-1})\sigma(t, T_{N-1}) \cdot dW_t^{T_N},\ \text{(from Case: } N-1) \\
&= (1 + \delta_{N-1} L(t, T_{N-1})) \frac{\delta_{N-1} L(t, T_{N-1})}{1 + \delta_{N-1} L(t, T_{N-1})} \sigma(t, T_{N-1}) \cdot dW_t^{T_N} \\
&= \frac{P(t, T_{N-1})}{P(t, T_N)} \Sigma(t, T_{N-1}, T_N) \cdot dW_t^{T_N},
\end{aligned}
$$

for $t \in [0, T_{N-1}]$, where $\Sigma(\cdot, T_{N-1}, T_N)$ is an $\mathbb{R}^d$-valued bounded process defined by

$$
\Sigma(t, T_{N-1}, T_N) := \frac{\delta_{N-1} L(t, T_{N-1})}{1 + \delta_{N-1} L(t, T_{N-1})} \sigma(t, T_{N-1}).
$$

Hence, by Girsanov's theorem (*Thm.* 3.11), we may define a new measure $\mathbb{Q}^{T_{N-1}}$ ($\sim \mathbb{Q}^{T_N}$) on $(\Omega, \mathcal{F}_{T_{N-1}})$ by the density

$$
\frac{d\mathbb{Q}^{T_{N-1}}}{d\mathbb{Q}^{T_N}} = \mathcal{E}_{T_{N-1}} \left(\int \Sigma(\cdot, T_{N-1}, T_N) \cdot dW^{T_N} \right),
$$

such that the process $W^{T_{N-1}}$ defined by

$$
W_t^{T_{N-1}} := W_t^{T_N} - \int_0^t \Sigma(s, T_{N-1}, T_N) ds,\ t \in [0, T_{N-1}],
$$

is a d-dimensional $\mathbb{Q}^{T_{N-1}}$-Brownian motion. Now we define the forward LIBOR rate corresponding to maturity T_{N-2} as the process satisfying

$$
dL(t, T_{N-2}) = L(t, T_{N-2})\sigma(t, T_{N-2}) \cdot dW_t^{T_{N-1}},\ t \in [0, T_{N-2}],
$$

$$L(0, T_{N-2}) = \frac{1}{\delta_{N-2}} \left(\frac{P(0, T_{N-2})}{P(0, T_{N-1})} - 1 \right).$$

We repeat this procedure until we have defined the forward LIBOR rate corresponding to expiry T_1 (Case: 1). The construction of the LIBOR market model is then complete, and for $0 \leq i < j \leq N-1$ the Brownian motions $W_t^{T_{i+1}}$ and $W_t^{T_{j+1}}$ are related through

$$\begin{aligned} dW_t^{T_{i+1}} &= dW_t^{T_{j+1}} - \sum_{k=i+1}^{j} \Sigma(t, T_k, T_{k+1}) dt \\ &= dW_t^{T_{j+1}} - \sum_{k=i+1}^{j} \frac{\delta_k L(t, T_k)}{1 + \delta_k L(t, T_k)} \sigma(t, T_k) dt. \end{aligned}$$

□

Corollary 3.19. *For $i = 1, ..., N-1$, the dynamics of $L(\cdot, T_i)$ under $\mathbb{Q}^{T_{k+1}}$ satisfy*

$$dL(t, T_i) = \begin{cases} L(t, T_i)\sigma(t, T_i) \cdot \left(-\sum_{j=i+1}^{k} \frac{\delta_j L(t, T_j)}{1+\delta_j L(t, T_j)} \sigma(t, T_j) dt + dW_t^{T_{k+1}} \right), & \text{if } i < k \\ L(t, T_i)\sigma(t, T_i) \cdot dW_t^{T_{i+1}}, & \text{if } i = k \\ L(t, T_i)\sigma(t, T_i) \cdot \left(\sum_{j=k+1}^{i} \frac{\delta_j L(t, T_j)}{1+\delta_j L(t, T_j)} \sigma(t, T_j) dt + dW_t^{T_{k+1}} \right), & \text{if } i > k, \end{cases}$$

for $t \in [0, T_i \wedge T_{k+1}]$, where $W^{T_{k+1}}$ is a d-dimensional $\mathbb{Q}^{T_{k+1}}$-Brownian motion.

We now introduce the discrete-money market process B^*, defined by Jamshidian (1997) as

$$B_t^* := P(t, T_{\eta(t)}) \prod_{i=0}^{\eta(t)-1} [1 + \delta_i L(T_i, T_i)], \ t \in [0, T_N], \tag{3.2}$$

where the function

$$\eta : [0, T_N] \to \{1, ..., N\}, \ t \mapsto \inf\{i \in \{1, ..., N\} \mid t < T_i\} \wedge N,$$

is the index of the next alive tenor maturity at t. B_t^* represents the (accumulated) value up to time t of an initial investment of $B_0^* = 1$ that is rolled over in successive bonds. To be more precise, the spot LIBOR portfolio initially starts with $B_0^* = 1$ and invests it in $[1 + \delta_0 L(T_0, T_0)]$ units of the shortest maturity bond available. Thereafter, at each time point $t = T_n$, the portfolio invests the proceeds accumulated up to time t in $\prod_{i=0}^{n} [1 + \delta_i L(T_i, T_i)]$ units of the T_n-bond. Over the period $[T_n, T_{n+1})$, the holding in the T_n-bond is kept constant until the next tenor maturity T_{n+1}. The value process of B_t^* in (3.2) thus follows. It is not difficult to check that B^* is continuous, which should be intuitively obvious from the spot LIBOR portfolio.

Definition 3.20 (Spot LIBOR measure)**.** The *spot LIBOR measure* on $(\Omega, \mathcal{F})$, denoted by $\mathbb{Q}^*$, is the EMM associated with the discrete-money market process B^* as a numéraire.

Jamshidian (1997) took a different approach to Musiela and Rutkowski (1997) by considering the dynamics of the LIBOR market model under a single measure, namely the spot LIBOR measure. As an alternative to the derivation provided by Jamshidian (1997), which utilises the theory of compensators, here we provide a derivation that is much simpler and in the same vein as the forward measure approach of Musiela and Rutkowski (1997).

The following lemma will be required in our derivation to provide the appropriate Girsanov transformation. Again, we consider an alternative proof to Filipovic (2009) by applying Itô's formula directly.

Lemma 3.21 (Filipovic (2009))**.** *Fix $m, n \in \{1, ..., N\}$ such that $m < n$. Then the T_n-forward process of the T_m-bond satisfies*

$$\frac{P(t,T_m)}{P(t,T_n)} = \frac{P(0,T_m)}{P(0,T_n)}\mathcal{E}_t\left(\int \Sigma(\cdot, T_m, T_n)\cdot dW^{T_n}\right),$$

for $t \in [0, T_m \wedge T_n]$, where $\Sigma(\cdot, T_m, T_n)$ is defined by

$$\Sigma(t,T_m,T_n) := \sum_{i=m}^{n-1} \frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)}\sigma(t,T_i).$$

Proof. In the construction of the LIBOR market model (see proof of *Thm.* 3.18), we obtained the following useful relationships:
The dynamics of one-period forward processes satisfy

$$d\left(\frac{P(t,T_i)}{P(t,T_{i+1})}\right) = \frac{P(t,T_i)}{P(t,T_{i+1})}\Sigma(t,T_i,T_{i+1})\cdot dW_t^{T_{i+1}},$$

and the Brownian motions $W_t^{T_{i+1}}$ and $W_t^{T_n}$ are related through

$$dW_t^{T_{i+1}} = dW_t^{T_n} - \Sigma(t,T_{i+1},T_n)dt.$$

Hence, by Kunita-Watanabe (*Thm.* 2.14), the quadratic covariation between one-period forward processes satisfies

$$d\left\langle \frac{P(\cdot,T_i)}{P(\cdot,T_{i+1})}, \frac{P(\cdot,T_j)}{P(\cdot,T_{j+1})}\right\rangle_t = \frac{P(t,T_i)}{P(t,T_{i+1})}\frac{P(t,T_j)}{P(t,T_{j+1})}\Sigma(t,T_i,T_{i+1})\cdot\Sigma(t,T_j,T_{j+1})dt.$$

Since the T_n-forward process of the T_m-bond may be written as

$$\frac{P(t,T_m)}{P(t,T_n)} = \prod_{i=m}^{n-1}\frac{P(t,T_i)}{P(t,T_{i+1})},$$

we can apply Itô's formula (*Thm.* 2.15) on the right-hand side to yield

$$
\begin{aligned}
&d\left(\frac{P(t,T_m)}{P(t,T_n)}\right)\\
&= \frac{P(t,T_m)}{P(t,T_n)}\left(\sum_{i=m}^{n-1}\Sigma(t,T_i,T_{i+1})\cdot dW_t^{T_{i+1}} + \right.\\
&\left.\frac{1}{2}\sum_{m\le i\ne j\le n-1}\Sigma(t,T_i,T_{i+1})\cdot\Sigma(t,T_j,T_{j+1})dt\right)\\
&= \frac{P(t,T_m)}{P(t,T_n)}\left(\sum_{i=m}^{n-1}\Sigma(t,T_i,T_{i+1})\cdot dW_t^{T_n} - \sum_{i=m}^{n-1}\Sigma(t,T_i,T_{i+1})\cdot\Sigma(t,T_{i+1},T_n)dt + \right.\\
&\left.\frac{1}{2}\sum_{m\le i\ne j\le n-1}\Sigma(t,T_i,T_{i+1})\cdot\Sigma(t,T_j,T_{j+1})dt\right)\\
&= \frac{P(t,T_m)}{P(t,T_n)}\Sigma(t,T_m,T_n)\cdot dW_t^{T_n},
\end{aligned}
$$

where the last equality follows from

$$
\begin{aligned}
\sum_{i=m}^{n-1}\Sigma(t,T_i,T_{i+1})\cdot\Sigma(t,T_{i+1},T_n) &= \sum_{i=m}^{n-1}\sum_{j=i+1}^{n-1}\Sigma(t,T_i,T_{i+1})\cdot\Sigma(t,T_j,T_{j+1})\\
&= \frac{1}{2}\sum_{m\le i\ne j\le n-1}\Sigma(t,T_i,T_{i+1})\cdot\Sigma(t,T_j,T_{j+1}).
\end{aligned}
$$

The result then follows by Doléans theorem (*Thm.* 3.10). □

Filipovic (2009) actually presents a slightly more general version of *Lem.* 3.21 by considering the case $m > n$, but this is not necessary for our purposes. The Jamshidian (1997) LIBOR market model follows next.

Theorem 3.22 (Jamshidian (1997))**.** *Suppose the family $\{\sigma(\cdot,T_i)\}$ satisfies the conditions in Def. 3.17. Then for $i = 1,...,N-1$, the dynamics of $L(\cdot,T_i)$ under $\mathbb{Q}^*$ satisfy*

$$dL(t,T_i) = L(t,T_i)\sigma(t,T_i)\cdot\left(\sum_{j=\eta(t)}^{i}\frac{\delta_j L(t,T_j)}{1+\delta_j L(t,T_j)}\sigma(t,T_j)dt + dW_t^*\right),\ t\in[0,T_i],$$

where W^* is a d-dimensional $\mathbb{Q}^*$-Brownian motion.

Proof. The theorem will be proved if we can find the appropriate Girsanov transformation that moves us from $\mathbb{Q}^{T_N}$ to $\mathbb{Q}^*$. Motivated by this, we consider the dynamics of the deflated process

$$\frac{B_t^*}{P(t,T_N)} = \left(\prod_{i=0}^{\eta(t)-1}[1+\delta_i L(T_i,T_i)]\right)\frac{P(t,T_{\eta(t)})}{P(t,T_N)},$$

for $t \in [0, T_N]$. We would like to apply Itô's formula (*Thm.* 2.15) on the right-hand side, however the discontinuities of the two processes inhibit us. Instead, if we consider the dynamics over (T_j, T_{j+1}), for $j = 0, ..., N-1$, then $\prod_{i=0}^{\eta(t)-1}[1 + \delta_i L(T_i, T_i)]$ is constant over each interval and $\frac{P(t,T_{\eta(t)})}{P(t,T_N)}$ is continuous. Thus Itô's formula (*Thm.* 2.15) applies and we have for $t \in (T_j, T_{j+1})$,

$$\begin{aligned} d\left(\frac{B_t^*}{P(t,T_N)}\right) &= \prod_{i=0}^{\eta(t)-1}[1 + \delta_i L(T_i, T_i)]d\left(\frac{P(t,T_{\eta(t)})}{P(t,T_N)}\right) \\ &= \prod_{i=0}^{\eta(t)-1}[1 + \delta_i L(T_i, T_i)]\frac{P(t,T_{\eta(t)})}{P(t,T_N)}\Sigma(t, T_{\eta(t)}, T_N) \cdot dW_t^{T_N} \\ &= \frac{B_t^*}{P(t,T_N)}\Sigma(t, T_{\eta(t)}, T_N) \cdot dW_t^{T_N}, \end{aligned}$$

where the second equality holds by *Lem.* 3.21. Then by Doléans theorem (*Thm.* 3.10),

$$\frac{B_t^*}{P(t,T_N)} = \frac{B_0^*}{P(0,T_N)}\mathcal{E}_t\left(\int \Sigma(\cdot, T_{\eta(\cdot)}, T_N) \cdot dW^{T_N}\right),$$

for $t \in [0, T_N]\backslash\mathbb{T}$. But since $\frac{B_t^*}{P(t,T_N)}$ and the Doléans exponential are continuous, equality must in fact hold everywhere on $[0, T_N]$. Then by *Thm.* 3.15, the measure $\mathbb{Q}^*$ on $(\Omega, \mathcal{F})$ is defined by the density

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}^{T_N}} := \mathcal{E}_{T_N}\left(\int \Sigma(\cdot, T_{\eta(\cdot)}, T_N) \cdot dW^{T_N}\right),$$

and by Girsanov's theorem (*Thm.* 3.11), the process W^* defined by

$$W_t^* := W_t^{T_N} - \int_0^t \Sigma(s, T_{\eta(s)}, T_N)ds,\ t \in [0, T_N],$$

is a d-dimensional $\mathbb{Q}^*$-Brownian motion. Using the relationship between the Brownian motions $W_t^{T_{i+1}}$ and $W_t^{T_N}$ (see proof of *Thm.* 3.18), we may express W_t^* in terms of $W_t^{T_{i+1}}$ as follows

$$W_t^* = W_t^{T_{i+1}} - \int_0^t \Sigma(s, T_{\eta(s)}, T_{i+1})ds,\ t \in [0, T_{i+1}].$$

Now recall that the dynamics of $L(\cdot, T_i)$ under $\mathbb{Q}^{T_{i+1}}$ satisfy

$$dL(t, T_i) = L(t, T_i)\sigma(t, T_i) \cdot dW_t^{T_{i+1}},\ t \in [0, T_i].$$

Hence under $\mathbb{Q}^*$, the dynamics of $L(\cdot, T_i)$ satisfy

$$dL(t, T_i) = L(t, T_i)\sigma(t, T_i) \cdot \left(\Sigma(t, T_{\eta(t)}, T_{i+1})dt + dW_t^*\right),\ t \in [0, T_i].$$

□

3.3 Alternative Formulation: LIBOR Market Model

In this section we discuss an alternative formulation of the LIBOR market model in terms of instantaneous volatility and correlation. The advantage of this second formulation is that it allows for a simple rank reduction of the model by separating the volatility and correlation components of $\sigma(\cdot, T_i)$ (discussed in Section 5.2.2). The material discussed below is based on Brigo and Mercurio (2007).

First Formulation: Covariance Matrix

In the first formulation of the LIBOR market model above, we assumed that the evolution of forward LIBOR rates is modelled by d random factors that are represented by independent d-dimensional Brownian motions. We also introduced the family $\{\sigma(\cdot, T_i)\}$ of $\mathbb{R}^d$-valued bounded deterministic volatility functions, where $\sigma(\cdot, T_i)$ described the responsiveness of $L(\cdot, T_i)$ to the d random shocks. These random shocks may represent, for example, movements in the yield curve such as a change in slope or curvature (Rebonato, 2002). In this formulation, the dynamics of the forward LIBOR rate $L(\cdot, T_i)$ are defined to satisfy

$$dL(t, T_i) = L(t, T_i)\sigma(t, T_i) \cdot dW_t^{T_{i+1}},\ t \in [0, T_i],$$

where $W^{T_{i+1}}$ is a d-dimensional $\mathbb{Q}^{T_{i+1}}$-Brownian motion. Using the result of Kunita-Watanabe (*Thm.* 2.14), the quadratic variation of $\log L(\cdot, T_i)$ is

$$d\langle \log L(\cdot, T_i)\rangle_t = \|\sigma(t, T_i)\|^2 dt,$$

and the quadratic covariation between $\log L(\cdot, T_i)$ and $\log L(\cdot, T_j)$ is

$$d\langle \log L(\cdot, T_i), \log L(\cdot, T_j)\rangle_t = \sigma(t, T_i) \cdot \sigma(t, T_j) dt.$$

Hence, the family $\{\sigma(\cdot, T_i)\}$ contains essential information regarding the instantaneous volatility of each forward LIBOR rate as well as the instantaneous covariance, thus correlation, between all pairs. This will be made more precise in the second formulation of the LIBOR market model below, where the instantaneous volatility and correlation are modelled directly as two separate components.

Second Formulation: Volatility and Correlation

In this formulation, the dynamics of the forward LIBOR rate $L(\cdot, T_i)$ are defined to satisfy

$$dL(t, T_i) = L(t, T_i)\sigma_i(t)\, dW_t^{i+1},\ t \in [0, T_i],$$

where W^{i+1} is a $\mathbb{Q}^{T_{i+1}}$-Brownian motion such that the family $\{W^i\}$ has instantaneous correlation $\rho : \mathbb{R} \to \mathbb{R}^{(N-1)\times(N-1)}$, i.e.

$$d\langle W^i, W^j\rangle_t = \rho_{i,j}(t)dt,$$

and σ_i is a bounded deterministic instantaneous volatility function. Unlike the first formulation, σ_i and W^{i+1} are scalar-valued processes. Again, using the result of Kunita-Watanabe (*Thm.* 2.14), the quadratic variation of $\log L(\cdot, T_i)$ is

$$d\langle \log L(\cdot, T_i)\rangle_t = \sigma_i^2(t)dt,$$

and the quadratic covariation between $\log L(\cdot, T_i)$ and $\log L(\cdot, T_j)$ is

$$d\langle \log L(\cdot, T_i), \log L(\cdot, T_j)\rangle_t = \sigma_i(t)\sigma_j(t)\rho_{i,j}(t)dt.$$

Hence, comparing the quadratic variation and covariation terms, we see that the second formulation is consistent with the first if we set

$$\sigma_i(t) = \|\sigma(t, T_i)\|, \tag{3.3}$$

$$\rho_{i,j}(t) = \frac{\sigma(t, T_i)\cdot\sigma(t, T_j)}{\|\sigma(t, T_i)\|\|\sigma(t, T_j)\|}, \tag{3.4}$$

$$dW_t^{i+1} = \frac{\sigma(t, T_i)}{\|\sigma(t, T_i)\|}\cdot dW_t^{T_{i+1}}. \tag{3.5}$$

Now if we let $R : \mathbb{R} \to \mathbb{R}^{(N-1)\times d}$ be the mapping that maps t to the matrix $R(t)$ whose i-th row $r_i(t) = \frac{\sigma(t,T_i)}{\|\sigma(t,T_i)\|}$, then we obtain the following relations

$$\rho(t) = R(t)R(t)^\top, \tag{3.6}$$

$$\sigma(t, T_i) = \sigma_i(t)r_i(t). \tag{3.7}$$

The second formulation thus decomposes the volatility function $\sigma(t, T_i)$ into a product of the instantaneous volatility and a component of the instantaneous correlation. From (3.6) and (3.7), it follows that the dynamics of $L(\cdot, T_i)$ under $\mathbb{Q}^*$ satisfy

$$dL(t, T_i) = L(t, T_i)\sigma_i(t)\left(\sum_{j=\eta(t)}^{i}\frac{\delta_j L(t, T_j)}{1+\delta_j L(t, T_j)}\sigma_j(t)\rho_{i,j}(t)dt + r_i(t)\cdot dW_t^*\right),\ t\in[0, T_i], \tag{3.8}$$

where W^* is a d-dimensional $\mathbb{Q}^*$-Brownian motion.

Conversely, assume the second formulation of the LIBOR market model is specified with volatility structure $\{\sigma_i(\cdot)\}$ and correlation structure ρ. If there exists a decomposition $\rho(t) = R(t)R(t)^\top$, then the first formulation can be recovered by simply setting $\sigma(t, T_i) = \sigma_i(t)r_i(t)$, where $r_i(t)$ is the i-th row of $R(t)$. To see that such a decomposition exists, first note that since $\rho(t)$ is symmetric and positive semidefinite, it admits the spectral decomposition

$$\rho(t) = Q(t)\Lambda(t)Q(t)^\top, \tag{3.9}$$

where $Q(t)$ is a real-valued orthogonal matrix whose columns are given by the eigenvectors of $\rho(t)$, and $\Lambda(t)$ is a nonnegative diagonal matrix whose entries are the associated eigenvalues. Now if we let $\Lambda(t)^{\frac{1}{2}}$ denote the principal square root of $\Lambda(t)$, i.e. $\Lambda(t) = \Lambda(t)^{\frac{1}{2}}\Lambda(t)^{\frac{1}{2}}$, and define $R(t) := Q(t)\Lambda(t)^{\frac{1}{2}}$, then we obtain the following relations

$$\rho(t) = R(t)R(t)^\top, \tag{3.10}$$
$$\Lambda(t) = R(t)^\top R(t). \tag{3.11}$$

Hence, the two formulations of the LIBOR market model are equivalent. Depending on the context, it may be more convenient to work with one formulation over the other. The modelling of the instantaneous volatility and correlation structures will be discussed in more detail in Sections 5.1–5.2; in particular, we provide parametric forms that will be used extensively throughout our analysis in Chapter 6. Finally, we define two useful quantities; V_i and $C_{i,j}$.

$$\begin{aligned}
V_i(t,u) &:= \langle \log L(\cdot,T_i)\rangle_u - \langle \log L(\cdot,T_i)\rangle_t \\
&= \int_t^u \|\sigma(s,T_i)\|^2 ds \\
&= \int_t^u \sigma_i^2(s) ds, \\
C_{i,j}(t,u) &:= \langle \log L(\cdot,T_i), \log L(\cdot,T_j)\rangle_u - \langle \log L(\cdot,T_i), \log L(\cdot,T_j)\rangle_t \\
&= \int_t^u \sigma(s,T_i)\cdot\sigma(s,T_j) ds \\
&= \int_t^u \sigma_i(s)\sigma_j(s)\rho_{i,j} ds.
\end{aligned}$$

3.4 Conditional Expectations: LIBOR Market Model

In this section we evaluate various conditional expectations that will prove useful when we discuss stochastic interpolation schemes. We begin with the computation of the conditional expectation $\mathbb{E}^{\mathbb{Q}^{T_i}}[L(u,T_i) \mid \mathcal{F}_t]$ for $t \le u \le T_i$, due to Jamshidian (1997). We provide a slightly different proof by following the approach of Rutkowski (1998). For this we require the following lemma.

Lemma 3.23. *Let $X \in c\mathcal{M}_0$ satisfy Novikov's criterion. If the quadratic variation $\langle X \rangle$ is deterministic, then*

$$\mathbb{E}[\mathcal{E}_u(X)^2 \mid \mathcal{F}_t] = \mathcal{E}_t(X)^2 e^{\langle X \rangle_u - \langle X \rangle_t}, \ t \leq u.$$

Proof. First notice that $\mathcal{E}(X)^2 = \mathcal{E}(2X)e^{\langle X \rangle}$. Hence $\mathcal{E}(X)^2 e^{-\langle X \rangle}$ is a martingale since $2X$ also satisfies Novikov's criterion (*Thm.* 3.12); i.e. if $t \leq u$, then

$$\mathbb{E}[\mathcal{E}_u(X)^2 e^{-\langle X \rangle_u} \mid \mathcal{F}_t] = \mathcal{E}_t(X)^2 e^{-\langle X \rangle_t}.$$

But since $\langle X \rangle$ is deterministic, the term $e^{-\langle X \rangle_u}$ can be taken outside the conditional expectation to yield the result. □

Theorem 3.24 (Jamshidian (1997))**.** *Fix $i \in \{1, ..., N-1\}$ and let $t \leq u \leq T_i$. Then the conditional expectation of $L(\cdot, T_i)$ under $\mathbb{Q}^{T_i}$ is*

$$\mathbb{E}^{\mathbb{Q}^{T_i}}[L(u, T_i) \mid \mathcal{F}_t] = L(t, T_i) \left(\frac{1 + \delta_i L(t, T_i) e^{V_i(t,u)}}{1 + \delta_i L(t, T_i)} \right).$$

Proof. From Bayes' formula (*Thm.* 3.8) we have

$$\mathbb{E}^{\mathbb{Q}^{T_i}}[L(u, T_i) \mid \mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}\left[L(u, T_i) \frac{d\mathbb{Q}^{T_i}}{d\mathbb{Q}^{T_{i+1}}} \mid \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}\left[\frac{d\mathbb{Q}^{T_i}}{d\mathbb{Q}^{T_{i+1}}} \mid \mathcal{F}_t\right]}.$$

By *Thm.* 3.15, the density $\frac{d\mathbb{Q}^{T_i}}{d\mathbb{Q}^{T_{i+1}}}$ on $(\Omega, \mathcal{F}_{T_i})$ is given by

$$\frac{d\mathbb{Q}^{T_i}}{d\mathbb{Q}^{T_{i+1}}} = \frac{P(T_i, T_i)/P(0, T_i)}{P(T_i, T_{i+1})/P(0, T_{i+1})} = \frac{1 + \delta_i L(T_i, T_i)}{1 + \delta_i L(0, T_i)}.$$

Using this representation, the conditional expectation can be written as

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^{T_i}}[L(u, T_i) \mid \mathcal{F}_t] &= \frac{\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}[L(u, T_i)(1 + \delta_i L(T_i, T_i)) \mid \mathcal{F}_t]}{\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}[1 + \delta_i L(T_i, T_i) \mid \mathcal{F}_t]} \\
&= \frac{\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}\left[L(u, T_i)\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}[1 + \delta_i L(T_i, T_i) \mid \mathcal{F}_u] \mid \mathcal{F}_t\right]}{1 + \delta_i L(t, T_i)} \\
&= \frac{\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}[L(u, T_i)(1 + \delta_i L(u, T_i)) \mid \mathcal{F}_t]}{1 + \delta_i L(t, T_i)} \\
&= \frac{L(t, T_i) + \delta_i \mathbb{E}^{\mathbb{Q}^{T_{i+1}}}[L(u, T_i)^2 \mid \mathcal{F}_t]}{1 + \delta_i L(t, T_i)},
\end{aligned}$$

where the second equality holds by the *tower property of conditional expectation*. Now recall that the dynamics of $L(\cdot, T_i)$ under $\mathbb{Q}^{T_{i+1}}$ satisfy

$$dL(u, T_i) = L(u, T_i)\sigma(u, T_i) \cdot dW_u^{T_{i+1}}, \ u \in [0, T_i].$$

Then by Doléans theorem (*Thm.* 3.10),

$$L(u, T_i) = L(0, T_i)\mathcal{E}_u\left(\int \sigma(\cdot, T_i) \cdot dW^{T_{i+1}}\right), \ u \in [0, T_i].$$

Since the volatility function $\sigma(\cdot, T_i)$ is deterministic, it follows by Kunita-Watanabe (*Thm.* 2.14) that the process $X := \int \sigma(\cdot, T_i) \cdot dW_u^{T_{i+1}}$ has a deterministic quadratic variation $\langle X \rangle$ given by

$$\langle X \rangle_t = \int_0^t \|\sigma(s, T_i)\|^2 ds.$$

Hence, by *Lem.* 3.23,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}\left[L(u, T_i)^2 \mid \mathcal{F}_t\right] &= L(t, T_i)^2 e^{\int_t^u \|\sigma(s, T_i)\|^2 ds} \\ &= L(t, T_i)^2 e^{V_i(t,u)}.\end{aligned}$$

The result follows. □

Drift-freezing Approximation (Brigo and Mercurio, 2007)

For $i = 1, ..., N-1$, the dynamics of $L(\cdot, T_i)$ under the measures discussed in the previous section may be written more compactly as

$$dL(t, T_i) = L(t, T_i)\left(\mu(L(t), t, T_i)dt + \sigma(t, T_i) \cdot dZ_t\right), \ t \in [0, T_i],$$

where Z is a d-dimensional Brownian motion. For example, under the spot LIBOR measure $\mathbb{Q}^*$, the Brownian motion $Z = W^*$ and the drift μ is

$$\mu(L(t), t, T_i) = \sum_{j=\eta(t)}^{i} \frac{\delta_j L(t, T_j)}{1 + \delta_j L(t, T_j)} \sigma(t, T_i) \cdot \sigma(t, T_j).$$

The dependence of μ on the state $L(\cdot)$ makes it difficult to evaluate conditional expectations involving $L(\cdot, T_{i-1})$. As an example, consider the conditional expectation $\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}[L(u, T_{i-1}) \mid \mathcal{F}_t]$ for $t \leq u \leq T_{i-1}$. A neat way to circumvent this problem is to partially freeze the drift so that μ will be a deterministic function. More specifically, the dynamics of $L(\cdot, T_i)$ conditional on $\mathcal{F}_s$ are approximated by

$$dL(t, T_i) \approx L(t, T_i)\left(\mu(L(s), t, T_i)dt + \sigma(t, T_i) \cdot dZ_t\right), \ t > s \in [0, T_i]. \tag{3.12}$$

We will refer to approximation (3.12) as the drift-freezing approximation at time s.

The next lemma utilises this approximation by providing closed-form expressions that will be useful for computing the forward process (4.2) under the Beveridge-Joshi interpolation scheme. The result of *Lem.* 3.25 is provided, without proof, in Beveridge and Joshi (2012). We provide a proof below.

Lemma 3.25. *Fix $i \in \{1, ..., N-1\}$ and let $t \leq u \leq T_{i-1}$. Then, assuming the drift-freezing approximation, the conditional expectation of $L(\cdot, T_{i-1})$ under $\mathbb{Q}^{T_{i+1}}$ is*

$$\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}[L(u, T_{i-1}) \mid \mathcal{F}_t] \approx L(t, T_{i-1}) e^{-\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)} C_{i,i-1}(t,u)},$$

and the conditional expectation of the product $L(\cdot, T_{i-1})L(\cdot, T_i)$ under $\mathbb{Q}^{T_{i+1}}$ is

$$\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}[L(u, T_{i-1})L(u, T_i) \mid \mathcal{F}_t] \approx L(t, T_{i-1})L(t, T_i) e^{C_{i,i-1}(t,u)} e^{-\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)} C_{i,i-1}(t,u)}.$$

Proof. Recall that the dynamics of $L(\cdot, T_{i-1})$ under $\mathbb{Q}^{T_i}$ satisfy

$$\begin{aligned} dL(u, T_{i-1}) &= L(u, T_{i-1})\sigma(u, T_{i-1}) \cdot dW_u^{T_i}, \; u \in [0, T_{i-1}] \\ &= L(u, T_{i-1})\sigma(u, T_{i-1}) \cdot \left(dW_u^{T_{i+1}} - \frac{\delta_i L(u, T_i)}{1 + \delta_i L(u, T_i)} \sigma(u, T_i) \right) \\ &\approx L(u, T_{i-1})\sigma(u, T_{i-1}) \cdot \left(dW_u^{T_{i+1}} - \frac{\delta_i L(t, T_i)}{1 + \delta_i L(t, T_i)} \sigma(u, T_i) \right), \end{aligned}$$

where the last line follows from the drift-freezing approximation at time t. Then by Doléans theorem (*Thm.* 3.10),

$$\begin{aligned} &L(u, T_{i-1}) \\ &\approx L(t, T_{i-1}) \mathcal{E}_u \left(\int_t^{\cdot} \sigma(s, T_{i-1}) \cdot dW_s^{T_{i+1}} - \frac{\delta_i L(t, T_i)}{1 + \delta_i L(t, T_i)} \int_t^{\cdot} \sigma(s, T_{i-1}) \cdot \sigma(s, T_i) ds \right) \\ &= L(t, T_{i-1}) \mathcal{E}_u \left(\int_t^{\cdot} \sigma(s, T_{i-1}) \cdot dW_s^{T_{i+1}} \right) e^{-\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)} \int_t^u \sigma(s,T_{i-1}) \cdot \sigma(s,T_i) ds} \\ &= L(t, T_{i-1}) \mathcal{E}_u \left(\int_t^{\cdot} \sigma(s, T_{i-1}) \cdot dW_s^{T_{i+1}} \right) e^{-\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)} C_{i,i-1}(t,u)}, \end{aligned} \tag{3.13}$$

for $u \in [t, T_{i-1}]$, where the second equality follows since $\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)} \int_t^u \sigma(s, T_{i-1}) \cdot \sigma(s, T_i) ds$ is a finite variation process (in u). Since σ is a deterministic function, the term $\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)} C_{i,i-1}(t, u)$ is also deterministic (in u) and hence can be taken outside the following conditional expectation. The conditional expectation of $L(\cdot, T_{i-1})$ under $\mathbb{Q}^{T_{i+1}}$ is then

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{T_{i+1}}}[L(u, T_{i-1}) \mid \mathcal{F}_t] &\approx L(t, T_{i-1}) \mathbb{E}^{\mathbb{Q}^{T_{i+1}}} \left[\mathcal{E}_u \left(\int_t^{\cdot} \sigma(s, T_{i-1}) \cdot dW_s^{T_{i+1}} \right) \mid \mathcal{F}_t \right] \\ &\quad \times e^{-\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)} C_{i,i-1}(t,u)} \\ &= L(t, T_{i-1}) \mathcal{E}_t \left(\int_t^{\cdot} \sigma(s, T_{i-1}) \cdot dW_s^{T_{i+1}} \right) e^{-\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)} C_{i,i-1}(t,u)} \\ &= L(t, T_{i-1}) e^{-\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)} C_{i,i-1}(t,u)}. \end{aligned}$$

For the second conditional expectation, first note that by Doléans theorem (*Thm.* 3.10),

$$L(u, T_i) = L(t, T_i) \mathcal{E}_u \left(\int_t^{\cdot} \sigma(s, T_i) \cdot dW_s^{T_{i+1}} \right), \; u \in [t, T_i]. \tag{3.14}$$

Now by equations (3.13)–(3.14) and using the identity $\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X+Y)e^{\langle X,Y\rangle}$ for $X, Y \in c\mathcal{M}_0$, we have

$$
\begin{aligned}
L(u,T_{i-1})L(u,T_i) &\approx L(t,T_{i-1})L(t,T_i)\mathcal{E}_u\left(\int_t^{\cdot}[\sigma(s,T_{i-1})+\sigma(s,T_i)]\cdot dW_s^{T_{i+1}}\right) \\
&\quad \times e^{\int_t^u \sigma(s,T_{i-1})\cdot\sigma(s,T_i)ds} e^{-\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)}C_{i,i-1}(t,u)} \\
&= L(t,T_{i-1})L(t,T_i)\mathcal{E}_u\left(\int_t^{\cdot}[\sigma(s,T_{i-1})+\sigma(s,T_i)]\cdot dW_s^{T_{i+1}}\right) \\
&\quad \times e^{C_{i,i-1}(t,u)} e^{-\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)}C_{i,i-1}(t,u)}
\end{aligned}
$$

for $u \in [t, T_{i-1}]$. Arguing similarly as above, the conditional expectation of the product $L(\cdot,T_{i-1})L(\cdot,T_i)$ under $\mathbb{Q}^{T_{i+1}}$ is then

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}[L(u,T_{i-1})L(u,T_i) \mid \mathcal{F}_t] \\
&\approx L(t,T_{i-1})L(t,T_i)\mathbb{E}^{\mathbb{Q}^{T_{i+1}}}\left[\mathcal{E}_u\left(\int_t^{\cdot}[\sigma(s,T_{i-1})+\sigma(s,T_i)]\cdot dW_s^{T_{i+1}}\right) \mid \mathcal{F}_t\right] \\
&\times\ e^{C_{i,i-1}(t,u)} e^{-\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)}C_{i,i-1}(t,u)} \\
&= L(t,T_{i-1})L(t,T_i)\mathcal{E}_t\left(\int_t^{\cdot}[\sigma(s,T_{i-1})+\sigma(s,T_i)]\cdot dW_s^{T_{i+1}}\right) \\
&\times e^{C_{i,i-1}(t,u)} e^{-\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)}C_{i,i-1}(t,u)} \\
&= L(t,T_{i-1})L(t,T_i)e^{C_{i,i-1}(t,u)} e^{-\frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)}C_{i,i-1}(t,u)}.
\end{aligned}
$$

□

Chapter 4

Interpolation Schemes

In practice, the tenor structure $\mathbb{T}$ is typically set to match standard market tenors so that the LIBOR market model describes the evolution of observable interest rates. In this way, the LIBOR market model is then automatically calibrated to the market caplet prices. On the other hand, this means direct pricing of interest rate contingent claims with cashflow dates outside the tenor structure $\mathbb{T}$ will not be possible. A common way to deal with this problem is to simply apply a naive interpolation method to obtain interest rates corresponding to the cashflow dates. Note, however, that these naive methods have no regard for possible arbitrage opportunities introduced as a result of the interpolation. Instead, here we concern ourselves with arbitrage-free interpolation schemes.

Following the above discussion, we can now state our problem more precisely. Fix $t_1 < t_2 \in [0, T_N]$ and let $t \in [0, t_1]$ be the current time. The problem is to determine the evolution of $L(t, t_1, t_2)$ from the family $\{L(\cdot, T_i)\}$ in a manner that respects arbitrage relationships between the forward rates. Since t_1, t_2 are arbitrary, this problem can be equivalently stated as finding an arbitrage-free continuous-tenor extension of the discrete-tenor LIBOR market model. For uniform year fractions $\delta_i = \delta$, the latter formulation of the problem enables us to complete the forward curve $L(t, T, T + \delta)$, $T \in [0, T_{N-1}]$, given the input set $\{L(t, T_i)\}$.

Before discussing the various interpolation schemes, we first introduce some useful terminology. We refer to $P(t, T)$ as a tenor bond if $T \in \mathbb{T}$, and non-tenor bond otherwise. $P(t, T)$ is called a short-bond if it matures on or before the next tenor maturity, i.e. if $T \leq T_{\eta(t)}$, otherwise $P(t, T)$ is called a long-bond. The following theorem illustrates the importance of this classification in simplifying the interpolation problem.

Theorem 4.1 (Schlögl (2002))**.** *Suppose a discrete-tenor LIBOR market model has been specified. Then the interpolation of tenor short-bonds, $P(t, T_{\eta(t)})$, fully determines the continuous-tenor LIBOR market model.*

Proof. (Schlögl, 2002). Let $t < u \in [0, T_N]$. Then the time t value of a u-bond satisfies

$$P(t,u) = P(t, T_{\eta(t)}) \left(\prod_{i=\eta(t)}^{\eta(u)-1} [1 + \delta_i L(t, T_i)]^{-1} \right) \frac{P(t,u)}{P(t, T_{\eta(u)})}.$$

For all $t < u$, the short-bond $P(t, T_{\eta(t)})$ is defined by the interpolation scheme and the product $\prod_{i=\eta(t)}^{\eta(u)-1}[1 + \delta_i L(t, T_i)]^{-1}$ is observable at each time point t. What remains to be defined is the forward process on the right-hand side. But the no-arbitrage constraint forces this forward process to satisfy

$$\frac{P(t,u)}{P(t, T_{\eta(u)})} = \mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}} \left[\frac{1}{P(u, T_{\eta(u)})} \mid \mathcal{F}_t \right].$$

Hence, all bonds $P(\cdot, u)$ in the continuous-tenor model are fully determined by the discrete-tenor model with an interpolation scheme for tenor short-bonds. □

Thus, it suffices to specify a convenient interpolation of tenor short-bonds in which case the interpolated forward rates satisfy

$$L(t, t_1, t_2) = \frac{1}{\delta_{1,2}} \left\{ \frac{P(t, t_1)}{P(t, T_{\eta(t_1)})} \left(\prod_{i=\eta(t_1)}^{\eta(t_2)-1} [1 + \delta_i L(t, T_i)] \right) \frac{P(t, T_{\eta(t_2)})}{P(t, t_2)} - 1 \right\}, \tag{4.1}$$

for $t \leq t_1$, where $\delta_{1,2} := t_2 - t_1$ and

$$\frac{P(t,u)}{P(t, T_{\eta(u)})} = \mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}} \left[\frac{1}{P(u, T_{\eta(u)})} \mid \mathcal{F}_t \right], \tag{4.2}$$

for $t < u \in [0, T_N]$. An interpolation scheme that results in a tractable expression for (4.2), and thus (4.1), is desirable as it enables easy calculation of $L(t, t_1, t_2)$, making the scheme useful in practice. Werpachowski (2010) highlights that computing expression (4.2) will generally be difficult in practice. However, in the case when tenor short-bonds satisfy $P(t, T_{\eta(t)})^{-1} = p(t, L(T_{\eta(t)-1}, T_{\eta(t)-1}))$ for some admissible function p, Schlögl (2002) argues the conditional expectation in (4.2) can be computed numerically since $L(\cdot, T_{\eta(t)-1})$ is lognormal with deterministic volatility.

The specification of interpolation schemes should be accompanied by an appropriate set of criteria to compare the schemes with each other. Beveridge and Joshi (2012) suggest the following three properties are desirable for any interpolation scheme:

(i) No internal arbitrage: No arbitrages are introduced from trading non-tenor bonds.

(ii) Positivity: Interpolated forward rates are positive. This precludes any cash arbitrages and results in sensible interpolations.

(iii) Stochasticity: The bond process $P(t_1, t_2)$ has stochastic dynamics for all $t_1 < t_2 \in [0, T_N]$.

An additional desirable feature is considered in this dissertation, that being the smoothness (or continuity) of interpolated forward rates. The discussion on smoothness of the four interpolation schemes introduced below is postponed until Chapter 6, when we examine our numerical results.

For now, we divide our discussion of interpolation schemes into two parts; schemes with zero short-bond volatility and schemes with non-zero short-bond volatility. Note, this is equivalent to classifying short-bond dynamics as being either deterministic or stochastic. We begin by introducing two deterministic schemes; Piterbarg deterministic interpolation and Schlögl deterministic interpolation.

4.1 Piterbarg Deterministic Interpolation (PDI)

Piterbarg (2003) introduces a simple interpolation scheme by assuming forward LIBOR rates satisfy

$$L(t, t_1, t_2) = L(T_{\eta(t_1)-1}, T_{\eta(t_1)-1}), \tag{4.3}$$

for $t \in [T_{\eta(t_1)-1}, t_1]$, $t_2 \leq T_{\eta(t_1)}$. In other words, the PDI scheme fixes the forward rate applying over the short period $[t_1, t_2]$, $t_2 \leq T_{\eta(t_1)}$, to equal the forward rate applying over $[T_{\eta(t_2)-1}, T_{\eta(t_2)}]$. Thus, the time t value of a u-bond can be seen to satisfy

$$P(t, u) = \begin{cases} \left[1 + (u - t)L(T_{\eta(t)-1}, T_{\eta(t)-1})\right]^{-1} & \text{if } u \leq T_{\eta(t)} \\ P(t, T_{\eta(t)}) \left(\prod_{i=\eta(t)}^{\eta(u)-2} [1 + \delta_i L(t, T_i)]^{-1} \right) P(T_{\eta(u)-1}, u) & \text{if } u > T_{\eta(t)}. \end{cases}$$

This now determines the interpolated forward rate $L(t, t_1, t_2)$ for the cases $t < T_{\eta(t_1)-1}$ and $t_2 > T_{\eta(t_1)}$. Beveridge and Joshi (2012) construct a simple portfolio to show the PDI scheme admits internal arbitrages. They argue as follows. Let the current time $t < t_1 < t_2 < T_{\eta(t)}$. Then construct a zero-cost bond portfolio by shorting a unit of $P(t, t_2)$ and longing $\frac{P(t,t_2)}{P(t,t_1)}$ units of $P(t, t_1)$. The time t_1 value of the bond portfolio satisfies

$$\frac{P(t, t_2)}{P(t, t_1)} - P(t_1, t_2) > 0,$$

since $P(t, t_2) > P(t, t_1)P(t_1, t_2)$, which itself follows from the inequality $1 + (t_2 - t)L < [1 + (t_1 - t)L][1 + (t_2 - t_1)L]$ for $L > 0$. Hence, arbitrage opportunities exist. It is evident from (4.3) that the PDI scheme fails to capture any stochasticity for short-bonds. Beveridge and Joshi (2012) also provide a proof to demonstrate the PDI scheme satisfies positivity.

4.2 Schlögl Deterministic Interpolation (SDI)

The SDI scheme discounts tenor short-bonds using the last-expired forward LIBOR rate. That is, tenor short-bonds are interpolated as

$$P(t, T_{\eta(t)})^{-1} = 1 + (T_{\eta(t)} - t)L(T_{\eta(t)-1}, T_{\eta(t)-1}). \tag{4.4}$$

By *Thm.* 4.1, the interpolated forward rate $L(t, t_1, t_2)$ is then determined by formula (4.1). A closed-form expression of the forward process (4.2) is provided by the following theorem.

Theorem 4.2 (Schlögl (2002))**.** *If tenor short-bonds, $P(t, T_{\eta(t)})$, are defined by the SDI scheme, then the forward process* (4.2) *satisfies*

$$\frac{P(t,u)}{P(t, T_{\eta(u)})} = 1 + (T_{\eta(u)} - u)L(t, T_{\eta(u)-1}),$$

for $t < u \in [0, T_N]$.

Proof. (Schlögl, 2002). Let $t < u \in [0, T_N]$. Since the SDI scheme defines tenor short-bonds as

$$P(t, T_{\eta(t)})^{-1} = 1 + (T_{\eta(t)} - t)L(T_{\eta(t)-1}, T_{\eta(t)-1}),$$

the forward process (4.2) simplifies to

$$\begin{aligned}\frac{P(t,u)}{P(t, T_{\eta(u)})} &= 1 + (T_{\eta(u)} - u)\mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}}\left[L(T_{\eta(u)-1}, T_{\eta(u)-1}) \mid \mathcal{F}_t\right] \\ &= 1 + (T_{\eta(u)} - u)L(t, T_{\eta(u)-1}).\end{aligned}$$

□

Thus in the SDI scheme, computation of formula (4.1) simplifies to

$$L(t, t_1, t_2) = \frac{1}{\delta_{1,2}}\left\{\frac{1 + (T_{\eta(t_1)} - t_1)L(t, T_{\eta(t_1)-1})}{1 + (T_{\eta(t_2)} - t_2)L(t, T_{\eta(t_2)-1})}\left(\prod_{i=\eta(t_1)}^{\eta(t_2)-1}[1 + \delta_i L(t, T_i)]\right) - 1\right\}.$$

Thm. 4.1 ensures that the resulting continuous-tenor LIBOR market model is free of arbitrage opportunities, hence the SDI scheme admits no internal arbitrages. It is evident from (4.4) that the SDI scheme fails to capture any stochasticity for short-bonds.

It turns out that for some real world applications, assuming short-bonds evolve deterministically can lead to unsatisfactory results (Schlögl, 2002). Now we introduce two stochastic schemes; Schlögl stochastic interpolation and Beveridge-Joshi stochastic interpolation.

4.3 Schlögl Stochastic Interpolation (SSI)

The SSI scheme discounts tenor short-bonds using a convex combination of the last-expired and the next-to-expire forward LIBOR rates. That is, tenor short-bonds are interpolated as

$$P(t, T_{\eta(t)})^{-1} = 1 + (T_{\eta(t)} - t) \left(\alpha(t) L(T_{\eta(t)-1}, T_{\eta(t)-1}) + (1 - \alpha(t)) L(t, T_{\eta(t)}) \right), \quad (4.5)$$

where $\alpha(\cdot)$ is a user-defined function that satisfies

$$\lim_{\Delta \downarrow 0} \alpha(T_i + \Delta) = 1$$
$$\lim_{\Delta \uparrow \delta_i} \alpha(T_i + \Delta) = 0.$$

As noted by Beveridge and Joshi (2012), $\alpha(\cdot)$ controls the level of short bond volatility; a smaller value of $\alpha(t)$ leads to a greater dependence on $L(t, T_{\eta(t)})$, and thus a greater volatility in $P(t, T_{\eta(t)})$. Following the suggestion of Schlögl (2002), we define $\alpha(t) := \frac{T_{\eta(t)} - t}{T_{\eta(t)} - T_{\eta(t)-1}}$.

By *Thm.* 4.1, the interpolated forward rate $L(t, t_1, t_2)$ is then determined by formula (4.1). A closed-form expression of the forward process (4.2) is provided by the following theorem.

Theorem 4.3 (Schlögl (2002)). *If tenor short-bonds, $P(t, T_{\eta(t)})$, are defined by the SSI scheme, then the forward process* (4.2) *satisfies*

$$\frac{P(t,u)}{P(t, T_{\eta(u)})} =$$
$$1 + (T_{\eta(u)} - u) \left(\alpha(u) L(t, T_{\eta(u)-1}) + (1 - \alpha(u)) L(t, T_{\eta(u)}) g(t, u, L(t, T_{\eta(u)})) \right),$$

for $t < u \in [0, T_N]$, where

$$g(t, u, L(t, T_{\eta(u)})) := \frac{1 + \delta_{\eta(u)} L(t, T_{\eta(u)}) e^{V_{\eta(u)}(t,u)}}{1 + \delta_{\eta(u)} L(t, T_{\eta(u)})}.$$

Proof. (Schlögl, 2002). Let $t < u \in [0, T_N]$. Since the SSI scheme defines tenor short-bonds as

$$P(t, T_{\eta(t)})^{-1} = 1 + (T_{\eta(t)} - t) \left(\alpha(t) L(T_{\eta(t)-1}, T_{\eta(t)-1}) + (1 - \alpha(t)) L(t, T_{\eta(t)}) \right),$$

the forward process (4.2) simplifies to

$$\frac{P(t,u)}{P(t, T_{\eta(u)})} = 1 + (T_{\eta(u)} - u) \Big(\alpha(u) \mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}} \left[L(T_{\eta(u)-1}, T_{\eta(u)-1}) \mid \mathcal{F}_t \right] +$$
$$(1 - \alpha(u)) \mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}} \left[L(u, T_{\eta(u)}) \mid \mathcal{F}_t \right] \Big)$$

$$= 1 + (T_{\eta(u)} - u) \big(\alpha(u)L(t, T_{\eta(u)-1}) + (1 - \alpha(u))\mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}} \left[L(u, T_{\eta(u)}) \mid \mathcal{F}_t\right]\big).$$

From *Thm.* 3.24, the conditional expectation of $L(u, T_{\eta(u)})$ under $\mathbb{Q}^{T_{\eta(u)}}$ is

$$\mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}}[L(u, T_{\eta(u)}) \mid \mathcal{F}_t] = L(t, T_{\eta(u)}) \left(\frac{1 + \delta_{\eta(u)} L(t, T_{\eta(u)}) e^{V_{\eta(u)}(t,u)}}{1 + \delta_{\eta(u)} L(t, T_{\eta(u)})}\right)$$
$$= L(t, T_{\eta(u)}) g(t, u, L(t, T_{\eta(u)})).$$

The result follows. □

Thm. 4.1 ensures that the resulting continuous-tenor LIBOR market model is free of arbitrage opportunities, hence the SSI scheme admits no internal arbitrages. As evidenced in (4.5), the dependence of short-bonds on the next-to-expire forward LIBOR rate ensures they remain stochastic until their maturities. Beveridge and Joshi (2012) find that the SSI scheme fails to handle sharply increasing forward curves as it can lead to negative forward rates, and thus unrealistic interpolations.

4.4 Beveridge-Joshi Stochastic Interpolation (BJSI)

Similar to the SSI scheme, the BJSI scheme discounts tenor short-bonds using a convex combination of the last-expired and the scaled next-to-expire forward LIBOR rates. That is, tenor short-bonds are interpolated as

$$P(t, T_{\eta(t)})^{-1} = 1 + (T_{\eta(t)} - t) \big(\alpha(t)L(T_{\eta(t)-1}, T_{\eta(t)-1}) + (1 - \alpha(t))L(t, T_{\eta(t)})\beta(t)\big), \tag{4.6}$$

where $\alpha(\cdot)$ is defined in (4.5) and $\beta(\cdot)$ is the scaling factor

$$\beta(t) := \frac{L(T_{\eta(t)-1}, T_{\eta(t)-1})}{L(T_{\eta(t)-1}, T_{\eta(t)})}.$$

Beveridge and Joshi (2012) introduced the scaling factor $\beta(\cdot)$ to handle sharply increasing forward curves. Note that with this factor, the BJSI scheme places even more emphasis on the last-expired forward LIBOR rate $L(T_{\eta(t)-1}, T_{\eta(t)-1})$. We shall return to this point in Chapter 6.

By *Thm.* 4.1, the interpolated forward rate $L(t, t_1, t_2)$ is then determined by formula (4.1). A closed-form approximation of the forward process (4.2) is provided by the following theorem.

Theorem 4.4 (Beveridge and Joshi (2012)). *If tenor short-bonds, $P(t, T_{\eta(t)})$, are defined by the BJSI scheme, then, assuming the drift-freezing approximation, the forward process* (4.2) *satisfies*

$$\frac{P(t,u)}{P(t,T_{\eta(u)})} \approx$$
$$1 + (T_{\eta(u)} - u)\left(\alpha(u)L(t,T_{\eta(u)-1}) + (1-\alpha(u))L(t,T_{\eta(u)})h(t,u,L(t,T_{\eta(u)}))\right),$$

for $t < u \in [0, T_N]$, where

$$h(t,u,L(t,T_{\eta(u)})) := \frac{L(t,T_{\eta(u)-1})}{L(t\wedge T_{\eta(u)-1},T_{\eta(u)})}e^{\left\{-\frac{\delta_{\eta(u)}L(t,T_{\eta(u)})}{1+\delta_{\eta(u)}L(t,T_{\eta(u)})}C_{\eta(u),\eta(u)-1}(t,T_{\eta(u)-1})\right\}}$$
$$\times\left(\frac{1+\delta_{\eta(u)}L(t,T_{\eta(u)})e^{C_{\eta(u),\eta(u)-1}(t,T_{\eta(u)-1})+V_{\eta(u)}(t\vee T_{\eta(u)-1},u)}}{1+\delta_{\eta(u)}L(t,T_{\eta(u)})}\right).$$

Proof. (Beveridge and Joshi, 2012). Let $t < u \in [0, T_N]$. Since the BJSI scheme defines tenor short-bonds as

$$P(t,T_{\eta(t)})^{-1} = 1 + (T_{\eta(t)} - t)\left(\alpha(t)L(T_{\eta(t)-1},T_{\eta(t)-1}) + (1-\alpha(t))L(t,T_{\eta(t)})\beta(t)\right),$$

where

$$\beta(t) = \frac{L(T_{\eta(t)-1},T_{\eta(t)-1})}{L(T_{\eta(t)-1},T_{\eta(t)})},$$

the forward process (4.2) simplifies to

$$\begin{aligned}\frac{P(t,u)}{P(t,T_{\eta(u)})} &= 1 + (T_{\eta(u)} - u)\Big(\alpha(u)\mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}}\left[L(T_{\eta(u)-1},T_{\eta(u)-1}) \mid \mathcal{F}_t\right] + \\ &\qquad (1-\alpha(u))\mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}}\left[L(u,T_{\eta(u)})\beta(u) \mid \mathcal{F}_t\right]\Big) \\ &= 1 + (T_{\eta(u)} - u)\big(\alpha(u)L(t,T_{\eta(u)-1}) + \\ &\qquad (1-\alpha(u))\mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}}\left[L(u,T_{\eta(u)})\beta(u) \mid \mathcal{F}_t\right]\Big),\end{aligned}$$

The proof now proceeds in two parts; first we consider the case $t \geq T_{\eta(u)-1}$, and finally the case $t < T_{\eta(u)-1}$.

For $t \geq T_{\eta(u)-1}$, the conditional expectation simplifies to

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}}\left[L(u,T_{\eta(u)})\beta(u) \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}}\left[L(u,T_{\eta(u)})\frac{L(T_{\eta(u)-1},T_{\eta(u)-1})}{L(T_{\eta(u)-1},T_{\eta(u)})} \mid \mathcal{F}_t\right] \\ &= \frac{L(T_{\eta(u)-1},T_{\eta(u)-1})}{L(T_{\eta(u)-1},T_{\eta(u)})}\mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}}\left[L(u,T_{\eta(u)}) \mid \mathcal{F}_t\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{L(T_{\eta(u)-1}, T_{\eta(u)-1})}{L(T_{\eta(u)-1}, T_{\eta(u)})} L(t, T_{\eta(u)}) \left(\frac{1 + \delta_{\eta(u)} L(t, T_{\eta(u)}) e^{V_{\eta(u)}(t,u)}}{1 + \delta_{\eta(u)} L(t, T_{\eta(u)})} \right) \\
&= L(t, T_{\eta(u)}) h(t, u, L(t, T_{\eta(u)})),
\end{aligned}
$$

where the third equality follows by *Thm.* 3.24, and the last by by noticing that $L(t, T_{\eta(u)-1}) = L(T_{\eta(u)-1}, T_{\eta(u)-1})$, $L(t \wedge T_{\eta(u)-1}, T_{\eta(u)}) = L(T_{\eta(u)-1}, T_{\eta(u)})$ and $C_{\eta(u),\eta(u)-1}(t, T_{\eta(u)-1}) = 0$ for $t \geq T_{\eta(u)-1}$.

For $t < T_{\eta(u)-1}$, using the *tower property of conditional expectation*, we have

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}} \left[L(u, T_{\eta(u)}) \beta(u) \mid \mathcal{F}_t \right] \\
&= \mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}} \left[L(u, T_{\eta(u)}) \frac{L(T_{\eta(u)-1}, T_{\eta(u)-1})}{L(T_{\eta(u)-1}, T_{\eta(u)})} \mid \mathcal{F}_t \right] \\
&= \mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}} \left[\frac{L(T_{\eta(u)-1}, T_{\eta(u)-1})}{L(T_{\eta(u)-1}, T_{\eta(u)})} \mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}} \left[L(u, T_{\eta(u)}) \mid \mathcal{F}_{T_{\eta(u)-1}} \right] \mid \mathcal{F}_t \right] \\
&= \mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}} \left[L(T_{\eta(u)-1}, T_{\eta(u)-1}) \left(\frac{1 + \delta_{\eta(u)} L(T_{\eta(u)-1}, T_{\eta(u)}) e^{V_{\eta(u)}(T_{\eta(u)-1},u)}}{1 + \delta_{\eta(u)} L(T_{\eta(u)-1}, T_{\eta(u)})} \right) \mid \mathcal{F}_t \right] \\
&= \mathbb{E}^{\mathbb{Q}^{T_{\eta(u)+1}}} \left[L(T_{\eta(u)-1}, T_{\eta(u)-1}) \left(1 + \delta_{\eta(u)} L(T_{\eta(u)-1}, T_{\eta(u)}) e^{V_{\eta(u)}(T_{\eta(u)-1},u)} \right) \mid \mathcal{F}_t \right] \\
&\times \frac{1}{1 + \delta_{\eta(u)} L(t, T_{\eta(u)})}
\end{aligned}
$$

where the third equality follows by *Thm.* 3.24, and the last by Bayes' formula (*Thm.* 3.8). Since the volatility function $\sigma(\cdot, T_{\eta(u)})$ is deterministic, the term $V_{\eta(u)}(T_{\eta(u)-1}, u)$ is also deterministic (in u) and hence can be taken outside the conditional expectation. Now by *Lem.* 3.25, the conditional expectation satisfies

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^{T_{\eta(u)}}} \left[L(u, T_{\eta(u)}) \beta(u) \mid \mathcal{F}_t \right] \\
&\approx \Bigg(L(t, T_{\eta(u)-1}) e^{-\frac{\delta_{\eta(u)} L(t, T_{\eta(u)})}{1 + \delta_{\eta(u)} L(t, T_{\eta(u)})} C_{\eta(u),\eta(u)-1}(t, T_{\eta(u)-1})} + \delta_{\eta(u)} L(t, T_{\eta(u)-1}) L(t, T_{\eta(u)}) \\
&\times e^{C_{\eta(u),\eta(u)-1}(t, T_{\eta(u)-1})} e^{-\frac{\delta_{\eta(u)} L(t, T_{\eta(u)})}{1 + \delta_{\eta(u)} L(t, T_{\eta(u)})} C_{\eta(u),\eta(u)-1}(t, T_{\eta(u)-1})} e^{V_{\eta(u)}(T_{\eta(u)-1},u)} \Bigg) \\
&\times \frac{1}{1 + \delta_{\eta(u)} L(t, T_{\eta(u)})} \\
&= L(t, T_{\eta(u)-1}) e^{\left\{ -\frac{\delta_{\eta(u)} L(t, T_{\eta(u)})}{1 + \delta_{\eta(u)} L(t, T_{\eta(u)})} C_{\eta(u),\eta(u)-1}(t, T_{\eta(u)-1}) \right\}} \\
&\times \left(\frac{1 + \delta_{\eta(u)} L(t, T_{\eta(u)}) e^{C_{\eta(u),\eta(u)-1}(t, T_{\eta(u)-1}) + V_{\eta(u)}(T_{\eta(u)-1},u)}}{1 + \delta_{\eta(u)} L(t, T_{\eta(u)})} \right) \\
&= L(t, T_{\eta(u)}) h(t, u, L(t, T_{\eta(u)})).
\end{aligned}
$$

The result follows. □

Thm. 4.1 ensures that the resulting continuous-tenor LIBOR market model is free of arbitrage opportunities, hence the BJSI scheme admits no internal arbitrages. As evidenced in (4.6), the dependence of short-bonds on the next-to-expire forward LIBOR rate ensures they remain stochastic until their maturities.

Beveridge and Joshi (2012) provide a sufficient condition, in the form of an inequality, under which interpolated forward rates satisfy positivity in the BJSI scheme. We summarise this result in the following theorem.

Theorem 4.5 (Beveridge and Joshi (2012))**.** *Let $t < t_1 < t_2 \in [0, T_N]$. If $\delta_i = \delta$ for all $i \in \{0, ..., N-1\}$ and $t_2 - t_1 = \delta$, then the Beveridge-Joshi interpolation scheme satisfies positivity, i.e. $L(t, t_1, t_2) > 0$, whenever*

$$e^{V_{\eta(t_2)}(T_{\eta(t_2)-1}, t_2)} < \frac{\delta}{T_{\eta(t_2)} - t_2} + \\ L(t, T_{\eta(t_1)-1}) \left(\alpha(t_1) + (1 - \alpha(t_1)) \frac{L(t, T_{\eta(t_1)})}{L(t \wedge T_{\eta(t_1)-1}, T_{\eta(t_1)})} \right) \left(\frac{1}{L(t, T_{\eta(t_2)-1})} + \delta \right). \tag{4.7}$$

Proof. The proof can be found in Beveridge and Joshi (2012). □

The conditions of the above theorem can be relaxed slightly by allowing the year fractions δ_i to be all different, in which case the expiry t_1 and maturity t_2 can be any time points such that $t < t_1 < t_2 \in [0, T_N]$. This of course comes at the cost of more cumbersome notation. Beveridge and Joshi (2012) stress that inequality (4.7) will hold in virtually all circumstances encountered in practice.

Chapter 5

Volatility and Correlation Structure

5.1 Volatility Structure

We assume a parametric form for the volatility structure, where the instantaneous volatility σ is given by the following parametrisation proposed in Rebonato (2002):

$$\sigma_i(t) = [a + b(T_i - t)]\, e^{-c(T_i - t)} + d,\ t \leq T_i, \tag{5.1}$$

where a, b, c, and d are constant parameters. These parameters may be found by calibrating the LIBOR market model to the market caplet prices, but here we shall specify them exogenously. This parametrisation of the volatility structure has several desirable features; particularly financial and numerical advantages. The proposed parametric form is time-homogeneous (i.e. the term structure of volatilities has the same shape throughout time) since the instantaneous volatility $\sigma_i(t)$ depends on the current time t through the time to expiry $T_i - t$. Furthermore, it allows for a great deal of flexibility in that we can fit both decreasing curves and humped shapes of the instantaneous volatility (Rebonato, 2002). In addition to these qualitative features, (5.1) allows an easy computation of the integral

$$\int_t^u \sigma_i(s)\sigma_j(s)\, ds, \tag{5.2}$$

by providing a closed-form expression for $t \leq u$. We summarise this result in the following lemma.

Lemma 5.1. *Let p and α be polynomials defined by $p(x) = \sum_{i=0}^{n} a_i x^i$ and $\alpha(x) = bx + c$, respectively. Then*

$$\int p(x)\, e^{\alpha(x)}\, dx = q(x)\, e^{\alpha(x)} + D,$$

where $D \in \mathbb{R}$ *and*

$$q(x) = \sum_{i=0}^{n} \left(\sum_{k=0}^{n-i} a_{i+k} \, (-1)^k \, \frac{k!}{b^{k+1}} \binom{i+k}{k} \right) x^i.$$

Proof. The proof can be found in Appendix A. □

But since the product $\sigma_i(s)\sigma_j(s)$ can be written as a sum of terms of the form $p(s)\, e^{\alpha(s)}$, we can apply *Lem.* 5.1 to formula (5.1) to yield the expression

$$\begin{aligned} \int_t^u \sigma_i(s)\sigma_j(s)\, ds = \Bigg[& d^2 s + \left\{ \frac{b^2}{2c} s^2 - \left(\frac{b^2}{2c^2} + \frac{b^2(T_i+T_j)}{2c} + \frac{ab}{c} \right) s + \left(\frac{b^2}{4c^3} + \right.\right. \\ & \left.\left. \frac{b^2(T_i+T_j)}{4c^2} + \frac{b^2 T_i T_j}{2c} + \frac{ab(T_i+T_j)}{2c} + \frac{ab}{2c^2} + \frac{a^2}{2c} \right) \right\} e^{-c(T_i+T_j-2s)} \\ & + \frac{d}{c} \left(a + \frac{b}{c} + b(T_i - s) \right) e^{-c(T_i-s)} \\ & + \frac{d}{c} \left(a + \frac{b}{c} + b(T_j - s) \right) e^{-c(T_j-s)} \Bigg] \Bigg|_{s=t}^{s=u}. \end{aligned} \tag{5.3}$$

The usefulness of this integral will become clear in Section 5.2.1 below. A significant drawback of this parametrisation, highlighted in Brigo and Mercurio (2007), is its inability to jointly calibrate the LIBOR model to market cap and swaption prices. This remark is not so crucial for us since we ignore swaptions in our presentation.

Rebonato (2002) argues that for the instantaneous volatility (5.1) to have valid financial and economic interpretations, the following constraints on the parameter set $\{a, b, c, d\}$ must be satisfied:

(i) $a + d > 0$.

(ii) $d > 0$.

(iii) $c > 0$.

These constraints have quite intuitive explanations. Since $\lim_{t \to T_i} \sigma_i(t) = a + d$, then $a + d$ is the asymptotic instantaneous volatility of near expiry rates. Similarly, if $c > 0$, then $\lim_{T_i \to \infty} \sigma_i(t) = d$ so that d is the asymptotic instantaneous volatility of far away rates. But volatility can never be negative, hence the constraints follow.

5.2 Correlation Structure

Recall that $\rho(t)$ is a correlation matrix if it is symmetric, positive semidefinite and satisfies $|\rho_{i,j}(t)| \leq 1$ with ones on the main diagonal, i.e. $\rho_{i,i}(t) = 1$. The instantaneous correlation $\rho_{i,j}(t)$ between the forward LIBOR rates $L(t, T_i)$ and $L(t, T_j)$ is defined as

$$\rho_{i,j}(t) := \frac{d\langle \log L(\cdot, T_i), \log L(\cdot, T_j)\rangle_t}{\sqrt{d\langle \log L(\cdot, T_i)\rangle_t \, d\langle \log L(\cdot, T_j)\rangle_t}}.$$

Quite clearly this defines a valid correlation matrix $\rho(t) = (\rho_{i,j}(t))$, and is consistent with the correlation specification provided in (3.4). For convenience, in this dissertation we assume a time-independent correlation structure so that it suffices to specify a single correlation matrix ρ. In practice, ρ is often implied from the market swaption prices or estimated from historical data (Rebonato, 2002). Before introducing a parametric form for the correlation, we first discuss some financially desirable features that ρ ought to have in the context of the LIBOR market model.

Brigo and Mercurio (2007) argue that ρ should satisfy the following three desirable properties:

(i) $\rho_{i,j} \geq 0$ for all i, j, i.e. all pairs of forward LIBOR rates are positively correlated.

(ii) The mapping $i \mapsto \rho_{i,j}$ is decreasing in i whenever $i \geq j$, i.e. forward LIBOR rates with closer maturities correlate more strongly than those with further maturities.

(iii) The mapping $i \mapsto \rho_{i+k,i}$ is increasing in i for each fixed k, i.e. forward LIBOR rates in the long end of the LIBOR curve correlate more strongly than those in the short end.

The last property follows from the market observation that the LIBOR curve tends to flatten out and move in a more correlated way in the long end of the curve.

5.2.1 Full-Rank Parametrisation

A full-rank parametrisation of the instantaneous correlation specifies a functional form that results in ρ having full rank, i.e. such that $\mathrm{rank}(\rho) = N - 1$, and thus models the evolution of forward LIBOR rates using $d = N - 1$ random factors. This, combined with the fact that a correlation matrix is symmetric and has ones on the main diagonal, then means ρ is fully characterised by $\frac{1}{2}(N-1)(N-2)$ entries. Having such a high number of entries may be difficult to achieve in practice, to which Brigo and Mercurio (2007) propose various full-rank parametrisations based on a reduced number of parameters. Here we adopt the classical two-parameter exponential decreasing parametrisation that defines instantaneous correlations $\rho_{i,j}$ by

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)e^{-\beta|i-j|}, \; \beta \geq 0, \tag{5.4}$$

where ρ_∞ is the asymptotic correlation between far away rates, and β is a constant parameter. The resulting correlation matrix ρ is positive definite and satisfies all the desirable properties discussed above except property (iii), a notable drawback. Nevertheless, the simplicity and time independence of this correlation structure has several numerical advantages; it simplifies the simulation of the LIBOR market model and allows for the covariance elements $C_{i,j}(t,u)$ to be rewritten as

$$\begin{aligned} C_{i,j}(t,u) &:= \int_t^u \sigma_i(s)\sigma_j(s)\rho_{i,j}(s)\,ds \\ &= \rho_{i,j}\int_t^u \sigma_i(s)\sigma_j(s)\,ds, \end{aligned}$$

for $t \leq u$. Thus from (5.3), we obtain a closed-form expression for $C_{i,j}(t,u)$.

Note that even though ρ is now fully characterised by just two parameters, we still require $d = N-1$ random factors to model the evolution of forward LIBOR rates. Such a high number of factors is computationally taxing and not desirable (e.g. in Monte Carlo simulations), thus a reduction of the model's dimension to $d < N-1$ is necessary. A reduced-rank parametrisation of the correlation ρ is discussed in the next section.

5.2.2 Reduced-Rank Parametrisation

A reduced-rank parametrisation of a correlation matrix ρ specifies another correlation matrix $\tilde{\rho}$ that in some sense "approximates" ρ and is of lower rank $d < N-1$. The rank reduction technique presented here is the well-known normalised principal components analysis (PCA), following the approach of Brigo and Mercurio (2007).

Assume $\rho \in \mathbb{R}^{(N-1)\times(N-1)}$ is positive definite. Recall from (3.9) that ρ admits the spectral decomposition

$$\rho = Q\Lambda Q^\top,$$

where $Q \in \mathbb{R}^{(N-1)\times(N-1)}$ is an orthogonal matrix whose columns are given by the eigenvectors of ρ, and $\Lambda \in \mathbb{R}^{(N-1)\times(N-1)}$ is a positive diagonal matrix whose entries are the associated eigenvalues arranged in descending order. In other words, $Q = (v_1, ..., v_{N-1})$ and $\Lambda = \operatorname{diag}(\lambda_1, ..., \lambda_{N-1})$ where $\{(v_i, \lambda_i)\}$ are the associated eigenvector-eigenvalue pairs of ρ ordered such that $\lambda_i < \lambda_j$ whenever $i > j$.

Now define the matrices

$$\begin{aligned} Q^{(d)} &:= (v_1, ..., v_d) \in \mathbb{R}^{(N-1)\times d} \\ \Lambda^{(d)} &:= \operatorname{diag}(\lambda_1, ..., \lambda_d) \in \mathbb{R}^{d\times d} \end{aligned}$$

$$R^{(d)} := Q^{(d)}\Lambda^{(d)} \in \mathbb{R}^{(N-1)\times d}$$
$$\rho^{(d)} := R^{(d)}{R^{(d)}}^{\top} \in \mathbb{R}^{(N-1)\times(N-1)},$$

where $\rho^{(d)}$ is a candidate d-rank correlation matrix that approximates ρ. In fact, it turns out that $\rho^{(d)}$ is the best d-rank approximation of ρ in the Frobenius norm. However, even though $\rho^{(d)}$ is symmetric and positive semidefinite, it need not satisfy $|\rho_{i,j}|\leq 1,\ \rho_{i,i} = 1$. Thus, in general, $\rho^{(d)}$ will not be a valid correlation matrix. Instead, if we interpret $\rho^{(d)}$ as a covariance matrix, then we can normalise it to obtain the associated correlation matrix $\tilde{\rho}$, a valid d-rank approximation of ρ. We proceed by defining the matrices

$$S := \operatorname{diag}(\rho^{(d)})^{\frac{1}{2}} \in \mathbb{R}^{(N-1)\times(N-1)} \tag{5.5}$$
$$\tilde{R} := S^{-1}R^{(d)} \in \mathbb{R}^{(N-1)\times d} \tag{5.6}$$
$$\tilde{\rho} := \tilde{R}\tilde{R}^{\top} \in \mathbb{R}^{(N-1)\times(N-1)}, \tag{5.7}$$

where $\operatorname{diag}(\rho^{(d)})$ extracts the diagonal of $\rho^{(d)}$ and $(\cdot)^{\frac{1}{2}}$ denotes the principal square root. Note, because of the normalisation technique, we lose optimality of the d-rank approximation in the Frobenius norm.

The d-rank dynamics of $L(\cdot, T_i)$ under $\mathbb{Q}^*$ satisfy

$$dL(t,T_i) = L(t,T_i)\sigma_i(t)\left(\sum_{j=\eta(t)}^{i}\frac{\delta_j L(t,T_j)}{1+\delta_j L(t,T_j)}\sigma_j(t)\tilde{\rho}_{i,j}dt + \tilde{r}_i \cdot dW_t^*\right),\ t\in[0,T_i], \tag{5.8}$$

where W^* is a d-dimensional $\mathbb{Q}^*$-Brownian motion, and $\tilde{r}_i$ is the i-th row of $\tilde{R}$. From equations (5.5)–(5.7), the (i,j)-th entry of $\tilde{\rho}$ is

$$\tilde{\rho}_{i,j} = \frac{\rho_{i,j}^{(d)}}{\sqrt{\rho_{i,i}^{(d)}\ \rho_{j,j}^{(d)}}}.$$

5.3 Caplet Pricing: LIBOR Market Model

The pricing of caplets within the LIBOR market model is a relatively simple problem if an appropriate measure is chosen. Below we illustrate the consistency of Black's futures formula with the LIBOR framework by deriving a Black-like formula for caplet prices. Quite simply, a caplet is a call option on some forward LIBOR rate. A more precise definition follows.

Definition 5.2 (Caplet)**.** A *caplet* with expiry T, maturity U and strike rate K is a contingent claim that pays

$$\delta(T,U)\left(L(T,T,U)-K\right)^+,$$

at maturity U. The time $t \le U$ value of this contract is denoted by $\mathrm{Cpl}(t, T, U, K)$.

Note that for $t \in [T, U]$, $\mathrm{Cpl}(t, T, U, K)$ behaves similarly to the bond $P(t, U)$ if the caplet expires in the money. It can be shown that $L(\cdot, T_i)$ satisfies (see proof of *Thm.* 3.24)

$$\log L(u, T_i) = \log L(t, T_i) - \frac{1}{2}\int_t^u \|\sigma(s, T_i)\|^2 ds + \int_t^u \sigma(s, T_i) \cdot dW_s^{T_{i+1}},\ u > t \in [0, T_i].$$

Hence, by Kunita-Watanabe (2.14), $L(\cdot, T_i)$ is lognormal under $\mathbb{Q}^{T_{i+1}}$ with

$$\begin{aligned}
\log L(T_i, T_i) &\overset{Q^{T_{i+1}}}{\sim} \mathcal{N}\left(\log L(t, T_i) - \frac{1}{2}\int_t^{T_i} \|\sigma(s, T_i)\|^2 ds, \int_t^{T_i} \|\sigma(s, T_i)\|^2 ds\right) \\
&\overset{Q^{T_{i+1}}}{\sim} \mathcal{N}\left(\log L(t, T_i) - \frac{1}{2}\int_t^{T_i} \sigma_i^2(s) ds, \int_t^{T_i} \sigma_i^2(s) ds\right) \\
&\overset{Q^{T_{i+1}}}{\sim} \mathcal{N}\left(\log L(t, T_i) - \frac{1}{2} V_i(t, T_i), V_i(t, T_i)\right).
\end{aligned}$$

Now by martingale pricing, the time $t \le T_i$ value of the i-th caplet struck at K is

$$\begin{aligned}
\mathrm{Cpl}(t, T_i, T_{i+1}, K) &= P(t, T_{i+1}) \mathbb{E}^{\mathbb{Q}^{T_{i+1}}} \left[\delta_i \left(L(T_i, T_i) - K\right)^+ \mid \mathcal{F}_t\right] \\
&= \delta_i P(t, T_{i+1}) \left[L(t, T_i)\Phi(d_1) - K\Phi(d_2)\right],
\end{aligned}$$

where Φ is the standard normal CDF and

$$\begin{aligned}
d_1 &= \frac{\log(L(t, T_i)/K) + \frac{1}{2} V_i(t, T_i)}{\sqrt{V_i(t, T_i)}} \\
d_2 &= d_1 - \sqrt{V_i(t, T_i)}.
\end{aligned}$$

Let $\sigma_B(t, T_i)$ denote the market implied volatility of the i-th caplet at time $t \le T_i$, i.e. the volatility obtained by inverting Black's futures formula. Then the LIBOR market model is automatically calibrated to the i-th market caplet price at time t if

$$\int_t^{T_i} \sigma_i^2(s) ds = \sigma_B^2(t, T_i) T_i. \tag{5.9}$$

Identity (5.9) is an additional desirable feature that any instantaneous volatility σ_i must satisfy. Unless further constraints are imposed, (5.9) does not, however, uniquely determine the volatility structure.

5.4 Simulation: LIBOR Market Model

Simulation of the LIBOR market model can be achieved under the spot LIBOR measure or any forward measure. Glasserman and Zhao (2000) found that the pricing of caplets when simulating under the spot LIBOR measure results in a lower variance than simulating under forward measures. Glasserman (2004) provides an explanation for this phenomenon by considering the following deflated bond prices under the two measures $\mathbb{Q}^*$ and $\mathbb{Q}^{T_N}$ respectively:

$$\frac{P(t,T_i)}{B_t^*} = \prod_{j=0}^{i-1}[1+\delta_j L(t \wedge T_j, T_j)]^{-1} \tag{5.10}$$

$$\frac{P(t,T_i)}{P(t,T_N)} = \prod_{j=i}^{N-1}[1+\delta_j L(t,T_j)], \tag{5.11}$$

for $t \in [0, T_i]$. Notice that (5.10) is bounded between 0 and 1, while (5.11) is unbounded. Hence, since expressions (5.10)–(5.11) appear under the expectation when pricing the payoff of a contingent claim at time $t = T_i$, we expect Monte Carlo simulation to result in a lower variance using (5.10) compared to (5.11). For this reason, all simulations will be performed under the spot LIBOR measure.

Owing to the difficulty of exact simulation, we resort to the Euler discretisation scheme on the fixed simulation grid $\mathbf{t} : 0 = t_0 < t_1 < ... < t_M \leq T_{N-1}$. Though not necessary, we also include the tenor maturities T_i, $i = 1, ..., N-1$, within the simulation grid $\mathbf{t}$, i.e. $\mathbb{T}\backslash\{T_N\} \subset \mathbf{t}$, so that each forward LIBOR rate is evolved up to its expiry.

Unfortunately, a direct application of the Euler scheme to SDE (5.8) can result in negative forward rates (Glasserman (2004) provides a detailed discussion). A possible solution is to apply the Euler scheme to $\log L(t,T_i)$ instead of $L(t,T_i)$, yielding the discretisation

$$\begin{aligned} \hat{L}(t_{k+1},T_i) &= \hat{L}(t_k,T_i)e^{\left(\mu(\hat{L}(t_k),t_k,T_i)-\frac{1}{2}\sigma_i^2(t_k)\right)\Delta t_k+\sqrt{\Delta t_k}\sigma_i(t_k)\tilde{r}_i\cdot Z_{k+1}} \\ \hat{L}(t_0,T_i) &= L(t_0,T_i) = \frac{1}{\delta_i}\left(\frac{P(t_0,T_i)}{P(t_0,T_{i+1})}-1\right), \end{aligned} \tag{5.12}$$

where $\Delta t_k := t_{k+1} - t_k$, $\{Z_k\}$ is a family of independent $\mathcal{N}_d(\mathbf{0}, \mathbf{I})$ random vectors, and the drift

$$\mu(\hat{L}(t_k),t_k,T_i) = \sum_{j=\eta(t_k)}^{i} \frac{\delta_j \hat{L}(t_k,T_j)}{1+\delta_j\hat{L}(t_k,T_j)}\sigma_i(t_k)\sigma_j(t_k)\tilde{\rho}_{i,j}.$$

Here $\hat{L}(\cdot,T_i)$ is an approximation of $L(\cdot,T_i)$ on the simulation grid $\mathbf{t}$. The Euler scheme (5.12) assumes that for $t \in [t_k, t_{k+1})$,

$$\sigma_i(t) \approx \sigma_i(t_k) \tag{5.13}$$

$$L(t,T_i) \approx L(t_k,T_i), \tag{5.14}$$

i.e. the drift and volatility are frozen to their initial values on each grid interval. Alternatively to (5.13), it is also possible to approximate $\sigma_i(t)$ by its root-mean-square over $[T_{\eta(t)-1}, T_{\eta(t)}]$, in which case

$$\sigma_i(t_k) \approx \sqrt{\frac{1}{T_{\eta(t_k)} - T_{\eta(t_k)-1}} \int_{T_{\eta(t_k)-1}}^{T_{\eta(t_k)}} \sigma_i^2(s)ds}.$$

The price $\mathrm{P}(\Psi)$ of a contingent claim Ψ with a payoff of $\Psi(L(t_k))$ at time t_k is approximated by

$$\begin{aligned} \mathrm{P}(\Psi) &= \mathbb{E}^{\mathbb{Q}^*}\left[\frac{\Psi(L(t_k))}{B^*_{t_k}}\right] \\ &\approx \frac{1}{n}\sum_{m=1}^{n} \frac{\Psi(\hat{L}_m(t_k))}{\hat{B}^*_m(t_k)}, \end{aligned} \tag{5.15}$$

where $\hat{L}_m(t_k)$ is the m-th (independent) simulation of the LIBOR market model up to time t_k and

$$\hat{B}^*_m(t_k) = \hat{P}_m(t_k, T_{\eta(t_k)}) \prod_{i=0}^{\eta(t_k)-1} [1 + \delta_i \hat{L}_m(T_i, T_i)]. \tag{5.16}$$

Since short-bonds are not uniquely determined in the LIBOR market model, (5.16) is generally only well-defined if an interpolation scheme has been specified. The justification of the Monte Carlo estimator (5.15) is provided by the strong law of large numbers (Glasserman, 2004).

Chapter 6

Numerical Results

In this section we illustrate the results obtained from implementing the various interpolation schemes using Matrix Laboratory (MATLAB, R2018b) software. The following parameters are fixed for all implementations of the LIBOR market model.

- Longest tenor maturity $T_N = 10yr$.
- Uniform year fractions $\delta_i = \delta := 0.25yr$ for all $i \in \{0, ..., N-1\}$. Thus $T_i = i\delta$.
- Simulate the LIBOR market model up to $t_M = T_{N-1}$, i.e. until expiry of the last forward LIBOR rate.
- Uniform simulation time steps $\Delta t_k = \Delta := 0.05yr$ for all $k \in \{0, ..., M-1\}$. Thus $t_k = k\Delta$.
- $d = 2$ random factors.
- $n = 5\,\text{million}$ fixed simulations.

In our results we consider two initial term structures of the LIBOR market model; the Vasiček ITS and ITS2 by Schlögl (2002).

Vasiček Initial Term Structure (ITS)

Vasiček ITS specifies the initial term structure of tenor bonds $P(\cdot, T_i)$ as

$$P(0, T_i) = e^{-A(T_i) - B(T_i) r_0},$$

for $i = 0, ..., N$, where r_0 is the initial short rate and

$$\begin{aligned} B(T_i) &:= \frac{1}{\gamma_1}\left(1 - e^{-\gamma_1 T_i}\right) \\ A(T_i) &:= \frac{\sigma_v^2 B^2(T_i)}{4\gamma_1} - \frac{(B(T_i) - T_i)\left(\gamma_1\gamma_2 - \frac{1}{2}\sigma_v^2\right)}{\gamma_1^2}, \end{aligned}$$

with the input parameter set $\{r_0, \gamma_1, \gamma_2, \sigma_v\}$ given by

Input Parameter Set			
r_0	γ_1	γ_2	σ_v
7%	0.15	0.09	2%

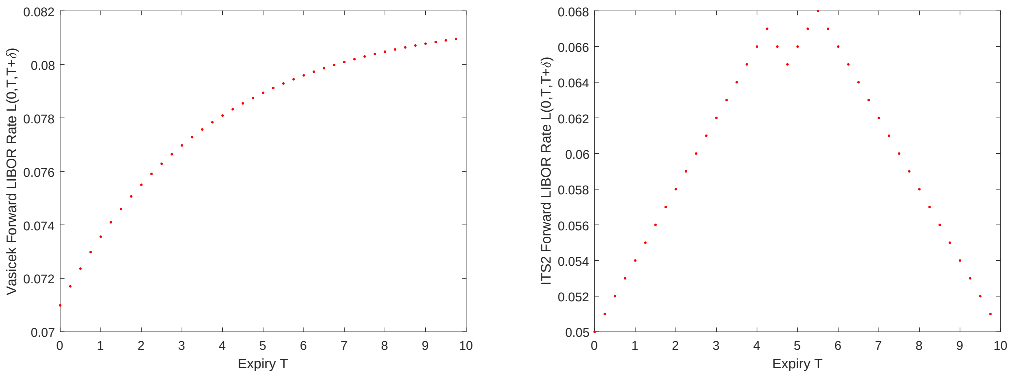
Tab. 6.1: Input parameters for the Vasiček Initial Term Structure.

Initial Term Structure 2 (ITS2) (Schlögl, 2002)

ITS2 specifies the initial term structure of forward LIBOR rates $L(\cdot, T_i)$ as

$$L(0,T_i) = \begin{cases} 5\% + 0.004(T_i - 0) & \text{if } T_i \in [0, 4.25] \\ 6.7\% - 0.004(T_i - 4.25) & \text{if } T_i \in [4.25, 4.75] \\ 6.5\% + 0.004(T_i - 4.75) & \text{if } T_i \in [4.75, 5.5] \\ 6.8\% - 0.004(T_i - 5.5) & \text{if } T_i \in [5.5, 10], \end{cases} \tag{6.1}$$

for $i = 0, ..., N-1$. Thus, $L(0, T_i)$ is piecewise linear on the coordinates set $\{(0, 5\%),$ $(4.25, 6.7\%), (4.75, 6.5\%), (5.5, 6.8\%), (10, 5\%)\}$.



(a) Vasiček ITS tenor forward curve. **(b)** ITS2 tenor forward curve.

Fig. 6.1: Initial Term Structures

The volatility (5.1) and correlation (5.4) structures are specified with parameters

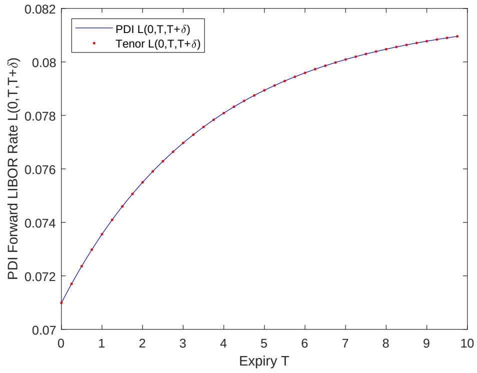
Input Parameter Set					
a	b	c	d	ρ_∞	β
0.04	0.09	0.44	0.15	0.5	0.05

Tab. 6.2: Input parameters for volatility and correlation structures.

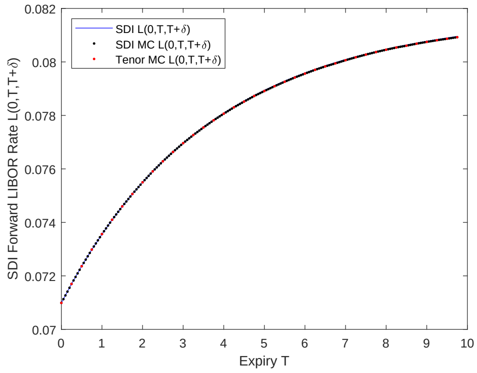
Appendices B & C provide plots for the corresponding volatility and correlation.

6.1 Forward Curves

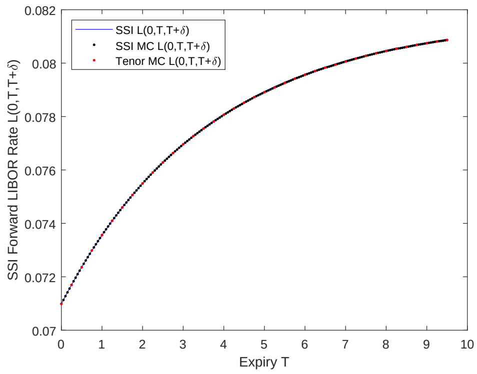
6.1.1 Vasiček Initial Term Structure (ITS)



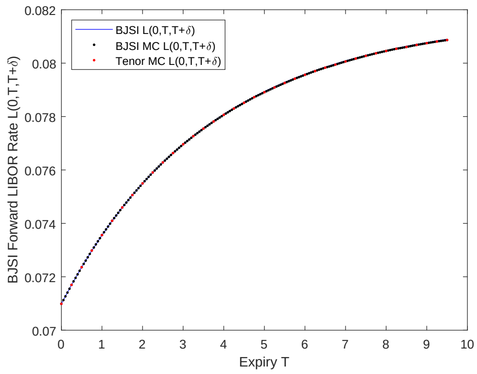
(a) PDI forward curve.



(b) SDI forward curve.



(c) SSI forward curve.

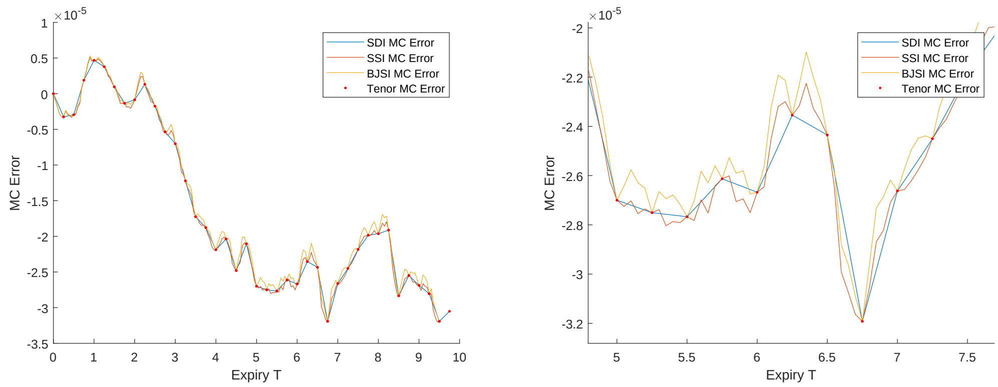


(d) BJSI forward curve.

Fig. 6.2: Interpolated forward curves (Vasiček ITS)

We assume the Vasiček initial term structure, whose forward LIBOR rates $L(0, T_i)$ are illustrated with red point markers in Figures 6.2a–6.2d. The blue solid line in each figure is constructed by directly applying each interpolation scheme to the tenor rates $\{L(0, T_i)\}$ to yield the interpolated forward curve $\{L(0, T, T + \delta) : T \in [0, T_N - \delta]\}$. The resulting forward curves appear identical, though they are not, since they must pass through the red point markers. The black point markers in Figures 6.2b–6.2d represent the Monte Carlo recovery of the interpolated forward curve under each scheme.

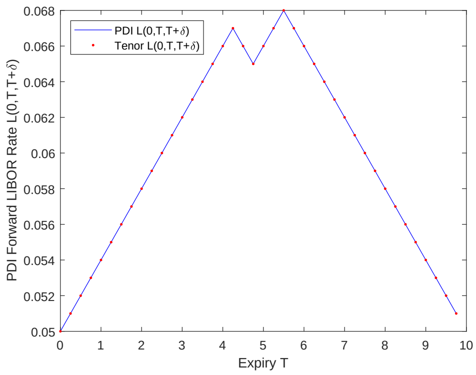
The presence of arbitrage in the PDI scheme implies no meaningful recovery can be achieved. For the SDI, SSI and BJSI schemes, we see an accurate Monte Carlo recovery of the interpolated forward curve. To distinguish between the recovery of these schemes, in Figure 6.3 we plot the resulting Monte Carlo error curve $\{\hat{L}(0, T, T + \delta) - L(0, T, T + \delta) : T \in [0, T_N - \delta]\}$ under each scheme. Figure 6.3a suggests that the Monte Carlo errors increase with expiry T, perhaps due to higher discretisation error as T increases. The SDI Monte Carlo error curve is approximately piecewise linear and smoother than the SSI and BJSI Monte Carlo error curves, a phenomenon we can attribute to the lack of short bond volatility in the SDI scheme. Figure 6.3b depicts the smoothness of the Monte Carlo error curves more clearly.



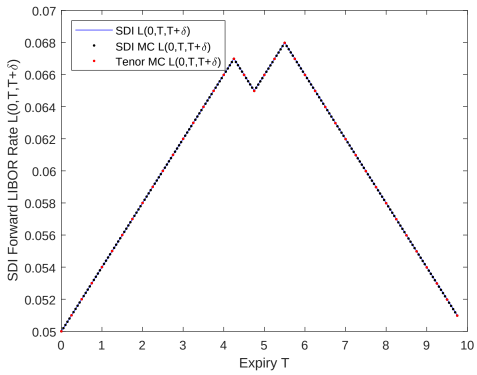
(a) MC recovery error curve. **(b)** MC recovery error curve.

Fig. 6.3: Error curves.

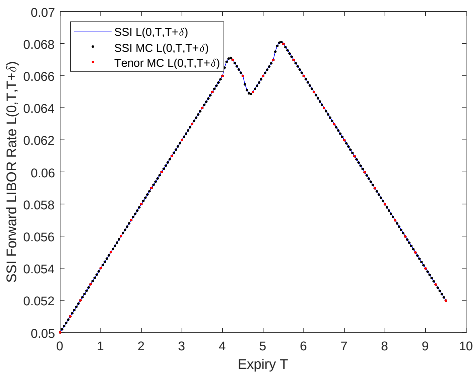
6.1.2 Initial Term Structure 2 (ITS2)



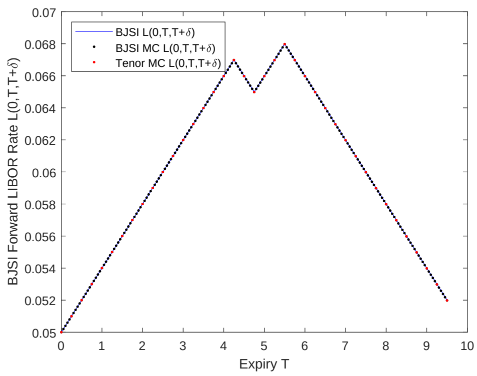
(a) PDI forward curve.



(b) SDI forward curve.



(c) SSI forward curve.



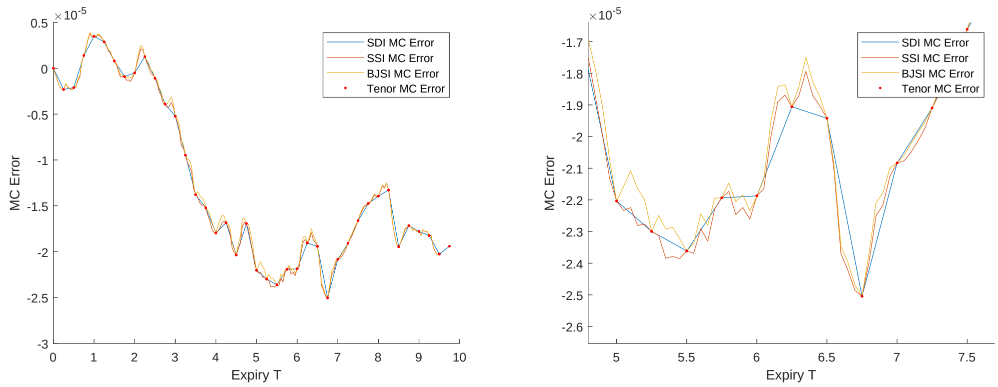
(d) BJSI forward curve.

Fig. 6.4: Interpolated forward curves (ITS2)

We assume the ITS2 initial term structure and reproduce the figures generated in the previous section for the Vasiček initial term structure. In this case, though, we see that the SSI forward curve is noticeably different at points corresponding to corners of the ITS2 graph, while the PDI, SDI and BJSI forward curves are similar. From Figures 6.4a, 6.4b & 6.4d, we see that the latter schemes apply an approximately linear interpolation to the tenor rates $\{L(0, T_i)\}$, resulting in very sharp corners where the ITS2 graph changes slope. By contrast, the SSI scheme smoothens the corners of the ITS2 graph by appropriately weighting the last-expired and the next-to-expire forward LIBOR rates. More generally, a closer look at formula (4.5) reveals that the SSI smoothing phenomenon occurs whenever the initial term structure changes slope abruptly.

Again, the presence of arbitrage in the PDI scheme implies no meaningful recovery can be achieved. For the SDI, SSI and BJSI schemes, we also see an accurate

Monte Carlo recovery of the interpolated forward curve. The resulting Monte Carlo error curves in Figure 6.5 are similar to those in the previous section for the Vasiček initial term structure. Accordingly, a similar analysis of the error curves also applies in this instance.



(a) MC recovery error curve. **(b)** MC recovery error curve.

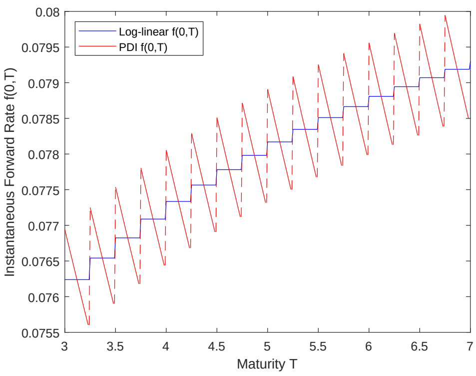
Fig. 6.5: Error curves.

6.2 Instantaneous Forward Curves

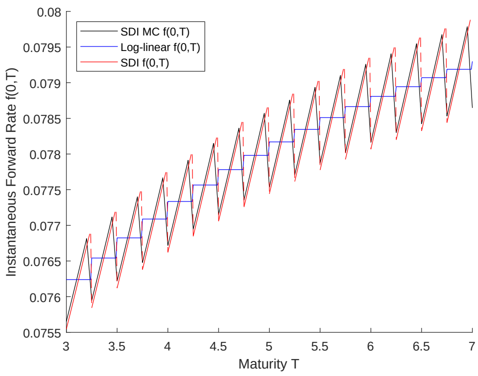
We saw in the previous section that for $\delta = 0.25yr$, the interpolated forward curve $\{L(0,T,T+\delta) : T \in [0, T_N - \delta]\}$ is well-behaved under each interpolation scheme. For $\delta \geq 0.25yr$, we still expect quite smooth forward curves, while it is unclear whether smoothness holds for smaller values of δ. To test this, we consider the extreme case when $\delta \to 0$ to obtain the instantaneous forward curve f. Numerically, we construct the instantaneous forward curve in the following manner. Let $\delta = 0.05yr$. Then the instantaneous forward rates are approximated by

$$f(0,T) \approx -\frac{1}{\delta}(\log P(0,T+\delta) - \log P(0,T))$$
$$\hat{f}(0,T) \approx -\frac{1}{\delta}(\log \hat{P}(0,T+\delta) - \log \hat{P}(0,T)).$$

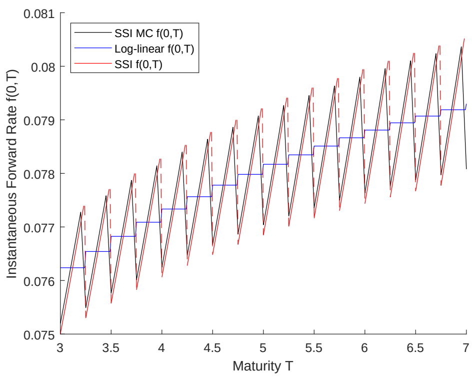
6.2.1 Vasiček Initial Term Structure (ITS)



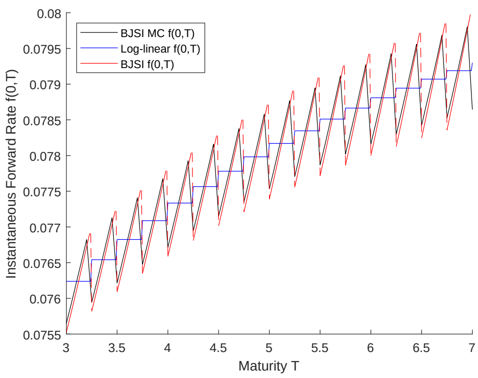
(a) PDI instantaneous forward curve.



(b) SDI instantaneous forward curve.



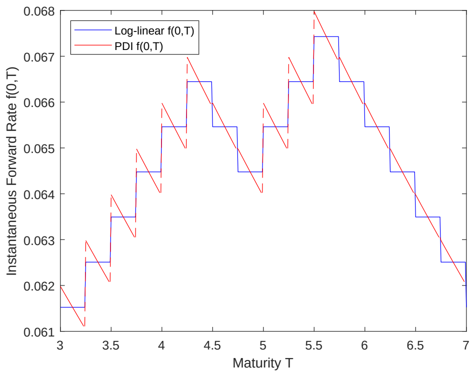
(c) SSI instantaneous forward curve.



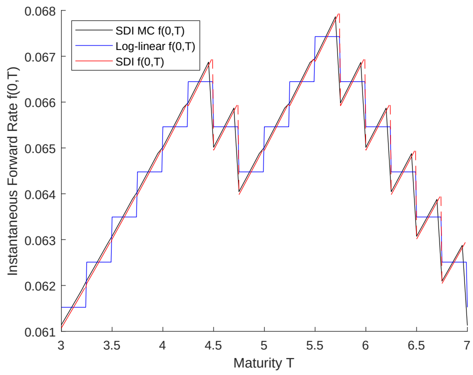
(d) BJSI instantaneous forward curve.

Fig. 6.6: Interpolated instantaneous forward curves (Vasiček ITS)

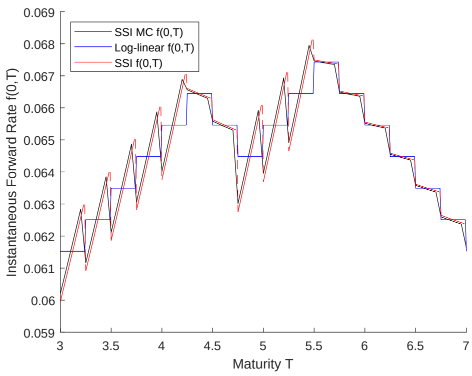
6.2.2 Initial Term Structure 2 (ITS2)



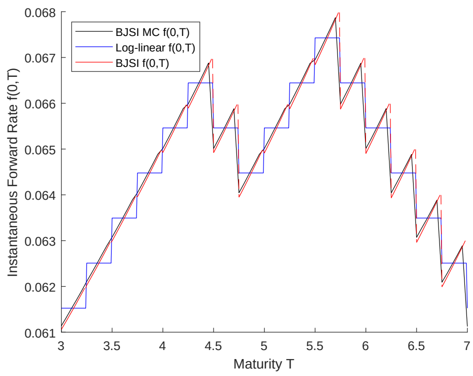
(a) PDI instantaneous forward curve.



(b) SDI instantaneous forward curve.



(c) SSI instantaneous forward curve.



(d) BJSI instantaneous forward curve.

Fig. 6.7: Interpolated instantaneous forward curves (ITS2)

Note, we compute $f(0,T)$ by directly applying each interpolation scheme to the initial term structure, while $\hat{f}(0,T)$ is computed as a Monte Carlo estimate of $f(0,T)$. In Figures 6.6-6.7 above, we plot the forward curves $\{f(0,T) : T \in [3,7]\}$ and $\{\hat{f}(0,T) : T \in [3,7]\}$ in red and black solid lines respectively. As a reference, we also plot the forward curves implied by log-linear interpolation of tenor bonds. The implied forward curves turn out to be piecewise constant, and can be seen from the blue staircase (or step) functions.

A striking feature of these plots is the sawtooth nature of the forward curves f and $\hat{f}$. From Figures 6.6-6.7, we see that the sawtooth pattern ramps downward for the PDI scheme, in contrast to the upward ramping sawtooth pattern observed for the SDI, SSI and BJSI schemes. The difference in ramps is a result of the PDI scheme fixing the forward rate applying over the short period $[T,U]$, $T_{\eta(U)-1} \le T \le T_{\eta(U)}$, to equal the forward rate applying over $[T_{\eta(U)-1}, T_{\eta(U)}]$. The jumps in the plots above correspond to maturities in the tenor structure $\mathbb{T}$, since reference rates (in a given scheme) are only changed when a separate accrual period is encountered.

The sawtooth nature is an unfortunate drawback of the schemes in question. An ideal interpolation scheme would be one that implies a smooth forward curve passing through each constant piece (e.g. midpoints) of the staircase function. Mathematically, a smooth fit can be achieved quite simply if no regard is made for arbitrage constraints. The requirement of no-arbitrage restricts the family of interpolation schemes deemed admissible, hence the difficulty in finding a wholly smooth arbitrage-free interpolation scheme.

The Monte Carlo forward curves $\hat{f}$ illustrate a successful recovery of the initial forward curves f in each scheme. A desirable property of these forward curves is that they pass through the midpoints of the respective staircase functions. To explain the similarity of SDI and BJSI forward curves, we compare the definitions of the two schemes. Since $\beta(t) = \frac{L(T_{\eta(t)-1}, T_{\eta(t)-1})}{L(T_{\eta(t)-1}, T_{\eta(t)})}$, we have the approximation

$$L(t, T_{\eta(t)})\beta(t) \approx L(T_{\eta(t)-1}, T_{\eta(t)-1}).$$

Thus, the SDI scheme is a deterministic approximation of the BJSI scheme. The error in this approximation is determined by the magnitude of the quantity $|L(t, T_{\eta(t)}) - L(T_{\eta(t)-1}, T_{\eta(t)})|$, and represents the presence of short bond volatility.

Chapter 7

Conclusion

In this dissertation we presented a comprehensive review of four interpolation schemes; namely the Piterbarg deterministic interpolation, Schlögl deterministic interpolation, Schlögl stochastic interpolation, and Beveridge-Joshi stochastic interpolation. These schemes were studied and compared with respect to the following three desirable properties suggested by Beveridge and Joshi (2012): (i) No internal arbitrage, (ii) Positivity, and (iii) Stochasticity. Furthermore, the smoothness implications of each scheme were thoroughly considered in the Numerical Results section.

We found that the Piterbarg scheme admitted internal arbitrage, satisfied positivity, and failed to capture any stochasticity for short bonds. The Schlögl deterministic scheme presented an improvement to the Piterbarg scheme by eliminating internal arbitrage, but it also failed to capture any stochasticity for short bonds. Schlögl (2002) remarked that lacking short-bond volatility can lead to unsatisfactory results in some real world applications. Considering this, Schlögl (2002) then introduced stochasticity to his deterministic scheme by attaching dependency on the next-to-expire forward LIBOR rate. Although the Schlögl stochastic scheme resolved the drawbacks of deterministic schemes, it possessed significant shortcomings of its own. Beveridge and Joshi (2012) noted that the Schlögl stochastic scheme failed to handle sharply increasing forward curves as it could lead to negative forward rates, hence unrealistic interpolations. The Beveridge-Joshi scheme then offered a partial solution by introducing a scaling factor that ensured positivity in virtually all practical circumstances.

The sawtooth nature of the implied instantaneous forward curves revealed a significant drawback of the four schemes. Nevertheless, considering that instantaneous forward rates are purely mathematical constructs, we cannot rule out the effectiveness of these interpolation schemes based on smoothness alone. Finally, consolidating the theoretical and numerical results above, we infer that the Beveridge-Joshi scheme is the most preferable interpolation method of the schemes reviewed.

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Appendix A

Proof of *Lem.* 5.1

Lemma A.1. *Let p and α be polynomials defined by $p(x) = \sum_{i=0}^{n} a_i x^i$ and $\alpha(x) = bx + c$, respectively. Then*

$$\int p(x)\, e^{\alpha(x)}\, dx = q(x)\, e^{\alpha(x)} + D,$$

where $D \in \mathbb{R}$ and

$$q(x) = \sum_{i=0}^{n} \left(\sum_{k=0}^{n-i} a_{i+k}\, (-1)^k \, \frac{k!}{b^{k+1}} \binom{i+k}{k} \right) x^i.$$

Proof. First we will show the result holds for the polynomial $p(x) = x^j$, where $j \in \mathbb{N}$. The general case will then follow.

$$\begin{aligned}
\int x^j\, e^{\alpha(x)}\, dx &= e^{\alpha(x)} \sum_{k=0}^{j} \frac{(-1)^k}{b^{k+1}} \, \frac{d^k}{dx^k}(x^j) + D_j \\
&= e^{\alpha(x)} \sum_{k=0}^{j} \frac{(-1)^k}{b^{k+1}} \, \frac{j!}{(j-k)!}\, x^{j-k} + D_j \\
&= e^{\alpha(x)} \sum_{k=0}^{j} (-1)^k \, \frac{k!}{b^{k+1}} \binom{j}{k}\, x^{j-k} + D_j,
\end{aligned}$$

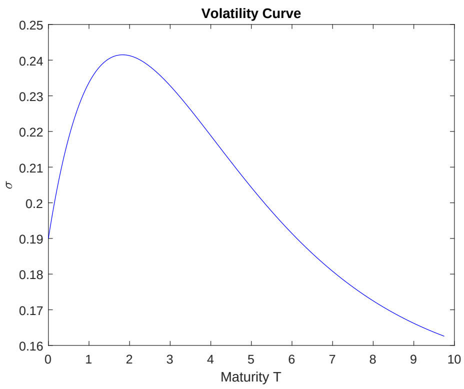
where $D_j \in \mathbb{R}$ and $\frac{d^0}{dx^0}(x^j) := x^j$. The first equality can be verified directly by differentiating the right-hand side to yield the integrand on the left-hand side. Now,

$$\begin{aligned}
\int p(x)\, e^{\alpha(x)}\, dx &= \sum_{j=0}^{n} a_j \int x^j\, e^{\alpha(x)}\, dx \\
&= e^{\alpha(x)} \sum_{j=0}^{n} \sum_{k=0}^{j} a_j\, (-1)^k \, \frac{k!}{b^{k+1}} \binom{j}{k}\, x^{j-k} + D \\
&= e^{\alpha(x)} \sum_{k=0}^{n} \sum_{j=k}^{n} a_j\, (-1)^k \, \frac{k!}{b^{k+1}} \binom{j}{k}\, x^{j-k} + D \\
&= e^{\alpha(x)} \sum_{i=0}^{n} \left(\sum_{k=0}^{n-i} a_{i+k}\, (-1)^k \, \frac{k!}{b^{k+1}} \binom{i+k}{k} \right) x^i + D,
\end{aligned}$$

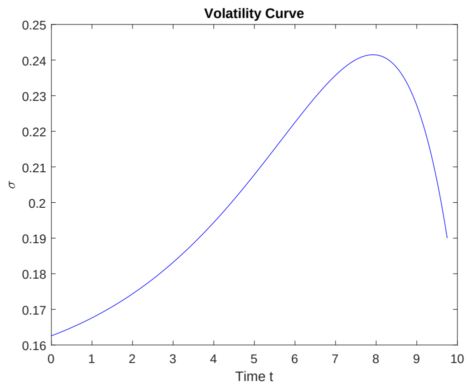
where $D = \sum_{j=0}^{n} a_j D_j \in \mathbb{R}$. $\square$

Appendix B

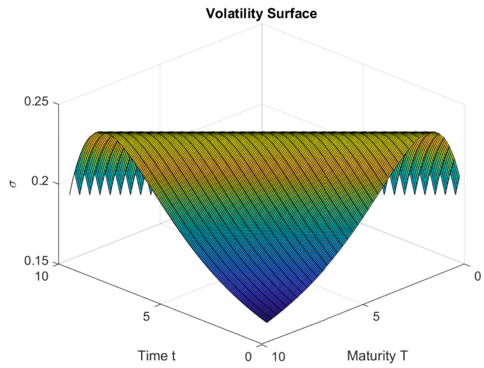
Volatility Structure



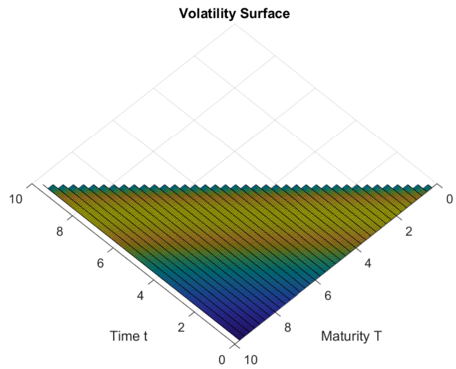
(a) Volatility curve plotting $\sigma_i(0)$ against T_i.



(b) Volatility curve plotting $\sigma_{9.75}(t)$ against t.



(c) Volatility surface plotting $\sigma_i(t)$ against (t, T_i).



(d) (Top-view) Volatility surface plotting $\sigma_i(t)$ against (t, T_i).

Fig. B.1: Volatility Structure

Note, Figures B.1a & B.1b are cross-sections of the volatility surface in Figure B.1c corresponding to the t-axis and T-axis respectively. The top-view in Figure B.1d illustrates that the volatility $\sigma_i(t)$ is not defined if $t > T_i$.

Appendix C

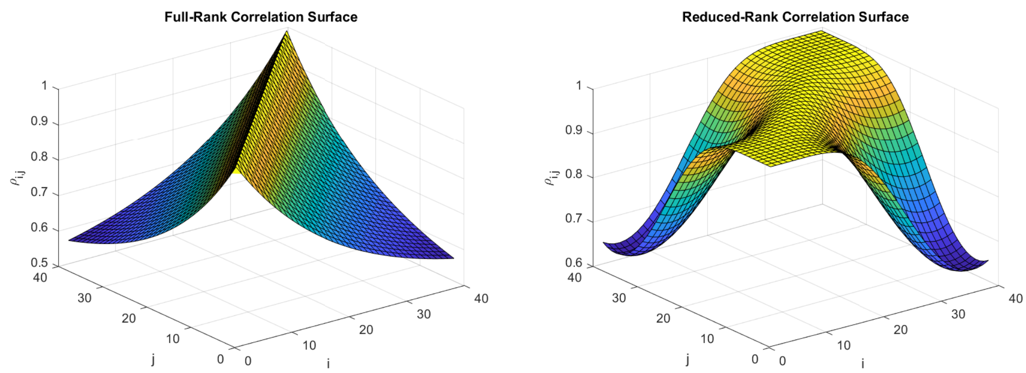
Correlation Structure

LIBOR Rates	$L(\cdot, 7.5)$	$L(\cdot, 7.75)$	$L(\cdot, 8)$	$L(\cdot, 8.25)$
$L(\cdot, 7.5)$	1	0.97561	0.95241	0.93035
$L(\cdot, 7.75)$	0.97561	1	0.97561	0.95241
$L(\cdot, 8)$	0.95241	0.97561	1	0.97561
$L(\cdot, 8.25)$	0.93035	0.95241	0.97561	1

Tab. C.1: Full-rank (parametrisation of) correlation matrix.

LIBOR Rates	$L(\cdot, 7.5)$	$L(\cdot, 7.75)$	$L(\cdot, 8)$	$L(\cdot, 8.25)$
$L(\cdot, 7.5)$	1	0.99977	0.99924	0.99860
$L(\cdot, 7.75)$	0.99977	1	0.99984	0.99949
$L(\cdot, 8)$	0.99924	0.99984	1	0.99990
$L(\cdot, 8.25)$	0.99860	0.99949	0.99990	1

Tab. C.2: Reduced-rank (parametrisation of) correlation matrix.



(a) Full-Rank Correlation Surface.

(b) Reduced-Rank Correlation Surface.

Fig. C.1: Correlation Structures

Recall the following two desirable properties of a correlation matrix ρ (Section 5.2): (i) $\rho_{i,j} \geq 0$ for all i, j, and (ii) the mapping $i \mapsto \rho_{i,j}$ is decreasing in i whenever $i \geq j$. Figure C.1a satisfies both properties. Figure C.1b satisfies (i), but only approximately satisfies (ii). That is, property (ii) holds everywhere except for (i, j) lying in the set $\{(x, y) : (x < 4, y > 36) \vee (x > 36, y < 4)\}$.

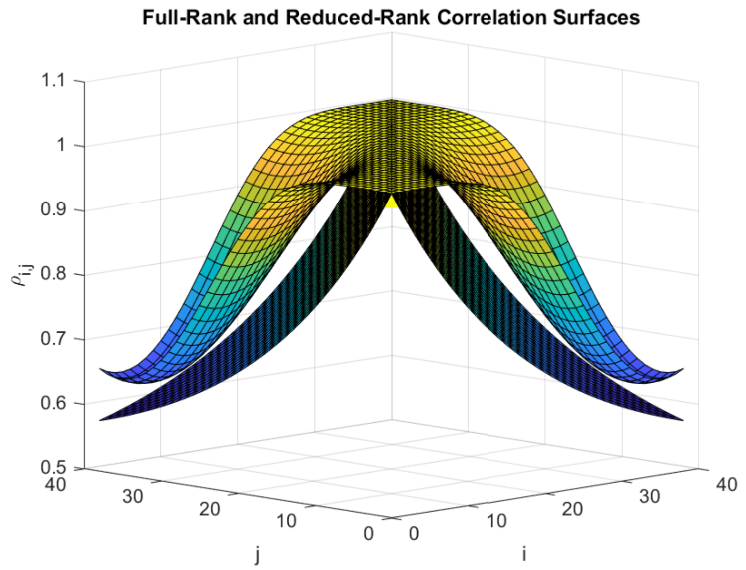


Fig. C.2: Full-Rank and Reduced-Rank Correlation Surfaces.

Note, the reduced-rank correlation surface lies above the full-rank correlation surface. Another feature to note is that the reduced-rank parametrisation delays the initial fall in correlations between neighbouring rates, especially in the long end of the surface. Tables C.1–C.2 illustrate this phenomenon for $30 \leq i, j \leq 33$.